Invertible Two Dimensional Topological Quantum Field Theories

Adrien Vakili

Supervisor: Anssi Lahtinen
University of Copenhagen

Colophon

This document was edited in `GNU Emacs` and typeset using LaTeX.
The text is set in Baskerville and the math is set in Computer Modern.

Abstract

The TQFTs are the symmetric monoidal functors from the cobordism category to some fixed target. When a TQFT factors through the cobordism groupoid, it is said to be invertible. In dimension two, Tillmann has shown that the cobordism groupoid is equivalent to an infinite cyclic group. As later remarked by Juer and Tillmann, a generator is given by the sphere. We provide a simplified version of Tillmann's proof, which makes this latter fact evident. As an application, we exhibit a symmetric monoidal equivalence between the category of invertible TQFTs and the group of automorphisms of the target's unit object. This generalizes a result of Juer and Tillmann.

Acknowledgements

During my studies in Copenhagen, I have had the privilege of meeting some outstanding members of the mathematics department. I would like to express my gratitude to Jesper Møller for his enthusiasm, Michal for his trust, Gijs and Beren for their stories, and Ernst Hansen for the interest he takes in the bachelor students.

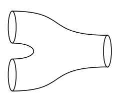
Thanks are due to my teachers Georgios, Elmiro, and Panos, who initiated me to mathematics. I am incredibly fortunate to have met them. In particular, I owe my taste for diagrams and for pictures to George and Panos, respectively.

I thank my friend Alvin, for empowering me with the tools that made the writing of this thesis a pleasure. I am also grateful to all my friends met at the University of Copenhagen for our fruitful conversations, and for their support. Thanks go to to Mossa, Hector, Martín, Leo, Morten, Tomas, Matthias, Ali, Corvin, Ajmal, Avgerinos, Zhipeng, Knut, Klaus, Cédric, and everyone who ever attended our Friday talks. Special thanks go to my dear friend Sriram, for riding at my side during the last three years. This thesis is dedicated to him.

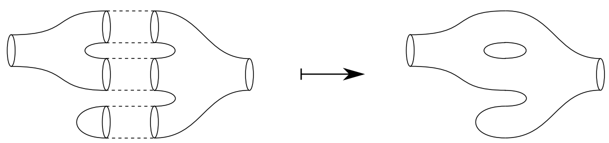
Contents

Introduction

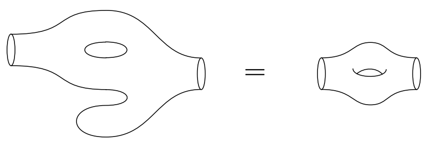
Our central object of study is the (two-dimensional) cobordism category $\mathcal{S}$. Roughly speaking, the objects of $\mathcal{S}$ are finite disjoint unions of circles, denoted $0, 1, 2, \ldots$, and the morphisms are (equivalence classes of) cobordisms between them. A cobordism from m to n is a (compact, orientable) surface Σ whose boundary decomposes into m "incoming" circles and n "outgoing" circles. An example of a cobordism from 2 to 1 can be depicted as follows:



Cobordisms can be composed by gluing along a matching boundary:



and two cobordisms are considered equivalent when they are homeomorphic relative to their boundary:



By the classification of surfaces, two connected cobordisms are equivalent if they have the same genus and the same numbers of incoming and outgoing boundary circles. In chapter 1, we rigorously define the cobordism category, and we provide an exposition of some results of Tillmann [8]. In particular, the classifying spaces of several subcategories of $\mathcal{S}$ are shown to be homotopy equivalent to the circle S^1.

These results may be taken as a hint that the classifying space of $\mathcal{S}$ is also homotopy equivalent to S^1, as conjectured by Juer and Tillmann in [5]. More evidence in favour of this conjecture is provided by the following. From a general result of Quillen [6], there is an isomorphism $\pi_1(\mathrm{B}\mathcal{S}) \cong \pi_1(\mathrm{B}\mathcal{G}\mathcal{S})$, where $\mathcal{G}\mathcal{S}$ is the *cobordism groupoid*, obtained from

$\mathcal{S}$ by formally inverting morphisms. In [8, Theorem 7], Tillmann proves that there is a equivalence of symmetric monoidal categories

$$\mathcal{GS} \simeq \mathbb{Z}, \tag{1}$$

where $\mathbb{Z}$ is the group of integers, seen as a one-object category. As a consequence, the fundamental group of the cobordism category is known:

$$\pi_1(\mathbf{B}\mathcal{S}) \cong \pi_1(\mathbf{B}\mathcal{GS}) \cong \pi_1(\mathbf{B}\mathbb{Z}) \cong \mathbb{Z}. \tag{2}$$

Juer and Tillmann [5] have remarked that the class of the sphere provides a generator for the infinite cyclic group in this equivalence. In theorem 2.12, we give simplified version of Tillmann's proof of eq. (1), which clearly exhibits the sphere as a generator.

The cobordism category is used to define *topological quantum field theories*, which have been of interest to both mathematicians and physicists. Formally, a $\mathcal{C}$-valued TQFT is a symmetric monoidal functor from the cobordism category to some arbitrary fixed target $\mathcal{C}$. Here the symmetric monoidal structure on the cobordism category is given by disjoint union. The TQFTs that factor through the groupoid $\mathcal{GS}$ are said to be *invertible*, and correspond to "anomaly theories" in physics. The invertible TQFTs can be arranged into a category $\mathbf{TQFT}^\times_{\mathcal{C}}$, which has been studied by Juer and Tillmann in [5]. In this paper, the authors consider the case when $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$ is the category of complex vector spaces (with tensor product as monoidal product). Specifically, using the fact that $\mathcal{GS} \simeq \mathbb{Z}$, they prove (in [5, Theorem 4.3]) that there is an equivalence of categories

$$\mathbf{TQFT}^\times_{\mathbf{Vect}_{\mathbb{C}}} \simeq \mathbb{C}^\times, \tag{3}$$

where $\mathbb{C}^\times$ is the discrete category of nonzero complex numbers. In theorem 3.6, we generalize eq. (3) to an equivalence

$$\mathbf{TQFT}^\times_{\mathcal{C}} \simeq \mathrm{Aut}_{\mathcal{C}}(e), \tag{4}$$

where $\mathcal{C}$ is any symmetric monoidal category, and e denotes its unit object. While Juer and Tillmann's result assumes that the TQFTs are *pointed* (i.e. preserve the unit object), we show that this assumption is unnecessary. Moreover, we are able to upgrade eq. (4) to a symmetric monoidal equivalence.

The results presented above rely on facts that are often stated without proof in the literature, and with good reason. Indeed, the arguments involved are usually straightforward but tedious. In an effort to make the material as self-contained and accessible as possible, we provide detailed proofs for a number of these facts. For instance, the precise statement of eq. (4) involves the symmetric monoidal structure on the groupoid $\mathcal{GS}$, which is carefully defined (in a general setting) in the first part of chapter 2.

In order to avoid interrupting the flow, some of the more laborious arguments have been relegated to the appendices. For instance, the proof of eq. (4) relies on the two following facts:

i) The small symmetric monoidal categories arrange into a 2-category, with symmetric monoidal variants of functors and natural transformations.

ii) Consequently, categories of symmetric monoidal functors are unchanged (up to equivalence) when replacing the source or target by an equivalent one.

Appendices A and B develop the basic theory needed to make these statements.

Structure of the text

The text is divided into three chapters and three appendices. The dependencies are roughly as follows:

$$\begin{tikzcd}
 & & \boxed{\text{1. The cobordism category}} \arrow[dl] \arrow[dr] & & \\
 & \boxed{\text{2. The cobordism groupoid}} & & \boxed{\text{3. Invertible TQFTs}} & \\
\boxed{\text{A. Strict 2-categories}} \arrow[ur] \arrow[rr] & & \boxed{\text{B. Symmetric monoidal gadgetry}} \arrow[ul] \arrow[ur] & & \boxed{\text{C. Categorical miscellanea}} \arrow[ul]
\end{tikzcd}$$

Chapter 1 introduces the cobordism category. The classifying spaces of some of its subcategories are computed, up to homotopy equivalence.
Chapter 2 defines the groupoid completion of a small category, and endows it with a symmetric monoidal structure when the small category is symmetric monoidal. The groupoid completion of the cobordism category is computed, up to symmetric monoidal equivalence.
Chapter 3 introduces invertible TQFTs and computes the category thereof, up to symmetric monoidal equivalence.
Appendix A examines equivalences in the context of 2-categories.
Appendix B presents the basic theory of symmetric monoidal categories
Appendix C collects various categorical facts.

Notation

- For x an object, the identity morphism of x is denoted 1_x.
- For $\mathcal{C}$ a category, the classifying space of $\mathcal{C}$ is denoted $\mathrm{B}\mathcal{C}$.

- The 2-category of small symmetric monoidal categories is denoted SM. For $\mathcal{C}, \mathcal{C}'$ small symmetric monoidal categories, the category of symmetric monoidal functors $\mathcal{C} \to \mathcal{C}'$ is denoted $\underline{\mathrm{SM}}(\mathcal{C}, \mathcal{C}')$.

1. The cobordism category

We introduce the notion of *(2-dimensional) cobordism*, and use it to define the *(2-dimensional) cobordism category* $\mathcal{S}$.

1.1. Oriented 2-cobordisms

Definition 1.1. Let M_0 and M_1 be closed (i.e. compact and without boundary) oriented 1-manifolds. An *oriented 2-cobordism* (or simply *cobordism*) from M_0 to M_1 consists of the following data:

- a compact oriented 2-manifold Σ with (possibly empty) boundary $\partial\Sigma$,
- a *decomposition* of the boundary $\partial\Sigma$ into closed 1-manifolds $\partial\Sigma_{\text{in}}$ and $\partial\Sigma_{\text{out}}$:
$$\partial\Sigma = \partial\Sigma_{\text{in}} \sqcup \partial\Sigma_{\text{out}}, \tag{1.1}$$
where $\partial\Sigma_{\text{in}}$ and $\partial\Sigma_{\text{out}}$ are called the *incoming* and *outgoing* boundaries, respectively,
- an orientation-preserving homeomorphism $M_0 \cong \partial\Sigma_{\text{in}}$, and an orientation-reversing homeomorphism $M_1 \cong \partial\Sigma_{\text{out}}$ Here the orientations on $\partial\Sigma_{\text{in}}$ and $\partial\Sigma_{\text{out}}$ are those induced by the orientation of Σ).

As a shorthand for the above data, we will write $\Sigma : M_0 \to M_1$, and say that Σ is a cobordism from M_0 to M_1. The extra data can be pictured in the following diagram:

$$\begin{tikzcd}
M_0 \arrow[d, "\cong"', "+"] & & M_1 \arrow[d, "\cong"', "-"] \\
\partial\Sigma_{\text{in}} \arrow[dr, hook] & & \partial\Sigma_{\text{out}} \arrow[dl, hook'] \\
& \Sigma, &
\end{tikzcd} \tag{1.2}$$

where the $+$ and $-$ signs indicate that the homeomorphisms are orientation-preserving and orientation-reversing, respectively. In the above situation, we sometimes abuse notation by identifying M_0 with $\partial\Sigma_{\text{in}}$ and M_1 with $\partial\Sigma_{\text{out}}$. In this case, we write the above diagram as

$$M_0 \xhookrightarrow{+} \Sigma \xhookleftarrow{-} M_1. \tag{1.3}$$

Definition 1.2. Two cobordisms Σ_0 and Σ_1 from M_0 to M_1 are said to be *equivalent* if there exists an orientation-preserving homeomorphism f commuting with the inclusions of M_0 and M_1, as in the following diagram:

$$\begin{tikzcd} & M_0 \arrow[dl, hook', bend right, "+"'] \arrow[dr, hook, bend left, "+"] & \\ \Sigma_0 \arrow[rr, "f", "\cong"'] & & \Sigma_1 . \\ & M_1 \arrow[ul, hook, bend left, "-"] \arrow[ur, hook', bend right, "-"'] & \end{tikzcd} \tag{1.4}$$

When this is the case, we say that f is a homeomorphism *relative to the boundaries*, and we write $\Sigma \sim \Sigma'$.

It is clear that $\sim$ is an equivalence relation.

Remark 1.3. Suppose that we modified the above definition in the following way: write $\Sigma \approx \Sigma'$ when there exists a homeomorphism $f : \Sigma \to \Sigma'$ as in eq. (1.4), but without the requirement that f be orientation-preserving. This turns out to be an equivalent definition. Clearly, if $\Sigma \sim \Sigma'$, then $\Sigma \approx \Sigma'$. To show the converse, suppose that $\Sigma \approx \Sigma'$. We consider two cases.

On the one hand, suppose that Σ is a closed surface (i.e. has no boundary components), and assume $\Sigma \approx \Sigma'$. If f is orientation-reversing, then f can be precomposed with an orientation-reversing self-homeomorphism $\Sigma \to \Sigma'$. Then gf is orientation-preserving, hence $\Sigma \sim \Sigma'$.

On the other hand, suppose that Σ has boundary components, and let $f : \Sigma \to \Sigma'$ be a homeomorphism. If f were orientation-reversing, then it would reverse the orientation of the boundary components. Then f would not make the diagram eq. (1.4) commute. Thus every homeomorphism $\Sigma \to \Sigma'$ is orientation-preserving, hence $\Sigma \sim \Sigma'$.

Composition of cobordisms

Equivalence classes of cobordisms will play the role of the morphisms in the (soon to be defined) cobordism category, and we must therefore have a way of composing them.

Definition 1.4. Let $\Sigma : M_0 \to M_1$ and $\Sigma' : M_1 \to M_2$, be cobordisms: The *composition* of Σ and Σ', denoted $\Sigma'\Sigma$, is defined to be the space obtained by the following pushout square in the category of topological manifolds:

$$\begin{tikzcd} M_1 \arrow[r, hook, "-"] \arrow[d, hook, "+"'] & \Sigma \arrow[d, hook] \\ \Sigma' \arrow[r, hook] & \Sigma'\Sigma . \end{tikzcd} \tag{1.5}$$

By abuse of notation, we identify Σ and Σ' with their images inside $\Sigma'\Sigma$.

Definition 1.4 gives us the first piece of data needed to define a cobordism $M_0 \to M_2$, namely that of a compact 2-manifold $\Sigma'\Sigma$ (with boundary). Once $\Sigma'\Sigma$ is oriented homeomorphisms $M_0 \xrightarrow{+} \partial\Sigma_{\text{in}}$ and $M_2 \xrightarrow{-} \partial\Sigma'_{\text{out}}$ will define homeomorphisms $M_0 \xrightarrow{+} \partial\Sigma'\Sigma_{\text{in}}$ and $M_2 \xrightarrow{-} \partial\Sigma'\Sigma_{\text{out}}$. In this way, the space $\Sigma'\Sigma$ can be equipped with the data of a cobordism $M_0 \to M_2$.

The next result shows to equip $\Sigma'\Sigma$ with an orientation.

Lemma 1.5. Let $\Sigma : M_0 \to M_1$ and $\Sigma' : M_1 \to M_2$, be cobordisms, and let $\Sigma'\Sigma$ be their composition. Then $\Sigma'\Sigma$ is orientable

Proof. To ease notation, we identify M_0, M_1 and M_2 with their images inside $\Sigma'\Sigma$. Moreover, in order to save space in diagrams, we write M_{ij} for $M_i \sqcup M_j$ (for $i, j \in \{ 0, 1, 2 \}$), and M_{012} for $M_0 \sqcup M_1 \sqcup M_2$. Observe that (under this abuse of notation), the boundary of $\Sigma'\Sigma$ is $\partial\Sigma'\Sigma = M_{02}$.

In order to orient $\Sigma'\Sigma$, we muse exhibit a fundamental class for $\Sigma'\Sigma$. Consider part of the long exact sequence for the triple $(\Sigma'\Sigma, M_{012}, M_{02})$:

$$0 \to H_2(\Sigma'\Sigma, M_{02}) \hookrightarrow H_2(\Sigma'\Sigma, M_{012}) \xrightarrow{\partial} H_1(M_{012}, M_{02}) \cong H_1(M_1). \tag{1.6}$$

Here the first term is $H_2(M_{012}, M_{02})$, which is 0 because M_{012} is 1-dimensional. The the last isomorphism is obtained by excision. This fits into the following commutative diagram:

$$\begin{tikzcd}
H_2(\Sigma, M_{01}) \oplus H_2(\Sigma', M_{12}) \arrow[r, "\partial"] \arrow[d, "\varphi"'] & H_1(M_{01}) \oplus H_1(M_{12}) \arrow[d] \\
H_2(\Sigma'\Sigma, M_{012}) \arrow[r, "\partial"] & H_1(M_{012}, M_{02}) \cong H_1(M_1),
\end{tikzcd} \tag{1.7}$$

where the vertical maps are sums of maps induced by inclusions, and the square commutes by the naturality of the connecting homomorphism ∂.

The composite ($\urcorner$) followed by the excision isomorphism is obtained by summing the following maps (where the last maps are given by projections).

$$\begin{aligned}
H_2(\Sigma, M_{01}) &\xrightarrow{\partial} H_1(M_{01}) \cong H_1(\partial\Sigma_{\text{in}}) \oplus H_1(\partial\Sigma_{\text{out}}) \to H_1(\partial\Sigma_{\text{out}}) \cong H_1(M_1), \\
H_2(\Sigma', M_{12}) &\xrightarrow{\partial} H_1(M_{12}) \cong H_1(\partial\Sigma'_{\text{in}}) \oplus H_1(\partial\Sigma'_{\text{out}}) \to H_1(\partial\Sigma'_{\text{in}}) \cong H_1(M_1),
\end{aligned} \tag{1.8}$$

Let $[\Sigma, M_{01}]$ and $[\Sigma', M_{01}]$ denote the fundamental classes of Σ and Σ', respectively, and let $[M_1]$ denote the fundamental class of M_1. Since $M_1 \to \partial\Sigma_{\text{out}}$ is orientation-reversing, the first map sends $[\Sigma, M_{01}]$ to $-[M_1]$, and since $M_1 \to \partial\Sigma'_{\text{in}}$ is orientation-preserving, the second map sends $[\Sigma', M_{12}]$ to $[M_1]$. Therefore the sum of these maps sends $([\Sigma, M_{01}], [\Sigma', M_{12}])$ to 0. By the commutativity of eq. (1.7) and the exactness of eq. (1.6), we conclude that $\varphi([\Sigma, M_{01}], [\Sigma', M_{12}])$ is a class in $H_2(\Sigma'\Sigma, M_{02})$. This class is our candidate for a fundamental class for $\Sigma'\Sigma$, so let us denote it $[\Sigma'\Sigma, M_{02}]$.

Let us verify that $[\Sigma'\Sigma, M_{02}]$ is indeed a fundamental class for $[\Sigma'\Sigma]$. We need to check that for each $x \in \operatorname{int}(\Sigma'\Sigma)$, the class $[\Sigma'\Sigma, M_{02}]$ restricts to a generator in $H_2(\Sigma'\Sigma, \Sigma'\Sigma - x)$. We consider three cases: either $x \in \operatorname{int}(\Sigma)$, or $x \in \operatorname{int}(\Sigma')$, or $x \in M_1$.

In the first case, consider the following diagram.

$$\begin{tikzcd}
 & H_2(\Sigma'\Sigma, M_{02}) \arrow[d] \arrow[dddd, dashed, bend left] \\
H_2(\Sigma, M_{01}) \oplus H_2(\Sigma', M_{12}) \arrow[r, "\varphi"] \arrow[d] \arrow[ddd, dashed, bend right] & H_2(\Sigma'\Sigma, M_{012}) \arrow[dd] \\
H_2(\Sigma, M_{01}) \oplus H_2(\Sigma', \Sigma') \arrow[d, "\cong"'] \arrow[dr] & \\
H_2(\Sigma, M_{01}) \arrow[r] \arrow[d] & H_2(\Sigma'\Sigma, M_0 \sqcup \Sigma') \arrow[d] \\
H_2(\Sigma, \Sigma - x) \arrow[r, "\cong"'] & H_2(\Sigma'\Sigma, \Sigma'\Sigma - x).
\end{tikzcd} \tag{1.9}$$

Here the vertical isomorphism is the projection, and the horizontal isomorphism is obtained by excision. The dashed maps are simply defined as the appropriate composites. All other maps are induced by inclusions, therefore the diagram commutes. We claim that the right-hand dashed map sends $[\Sigma'\Sigma, M_{02}]$ to a generator in $H_2(\Sigma'\Sigma, \Sigma'\Sigma - x)$. By definition, $[\Sigma'\Sigma, M_{02}] = \varphi([\Sigma, M_{01}], [\Sigma', M_{12}])$. Therefore, by commutativity of the diagram, it suffices to check that the composite ($\lrcorner$) sends $([\Sigma, M_{01}], [\Sigma', M_{12}])$ to a generator. But recall that $[\Sigma, M_{01}]$ is a fundamental class for Σ, so that the left-hand dashed map sends this class to a generator in $H_2(\Sigma, \Sigma - x)$. Since the bottom map is an isomorphism, this generator corresponds to a generator in $H_2(\Sigma'\Sigma, \Sigma'\Sigma - x)$, and we are done. In the second case, i.e. when $x \in \operatorname{int}(\Sigma')$, we argue similarly as when $x \in \operatorname{int}(\Sigma)$.

In the third case, we assume $x \in M_1$. By picking a chart around x, we can reduce the situation to the first case. More precisely, let $\psi : U \to \mathbb{R}^2$ be a chart around x, so that U is an open neighborhood of x, and ψ is a homeomorphism onto some open ball $V \subset \mathbb{R}^2$. Then there exists a point $y \in U$ which lies outside of M_1, say $y \in \operatorname{int}(\Sigma)$. Now consider the following diagram:

$$\begin{tikzcd}
H_2(\Sigma'\Sigma, M_{02}) \arrow[r] \arrow[d, dashed] & H_2(\Sigma'\Sigma, \Sigma'\Sigma - x) \\
H_2(\Sigma'\Sigma, \Sigma'\Sigma - y) \arrow[d, "\cong"'] & H_2(\mathbb{R}^2, \mathbb{R}^2 - \psi(x)) \arrow[u, "\cong"'] \\
H_2(\mathbb{R}^2, \mathbb{R}^2 - \psi(y)) \arrow[r, "\cong"'] & H_2(\mathbb{R}^2, \mathbb{R} - V) \arrow[u, "\cong"']
\end{tikzcd} \tag{1.10}$$

All isomorphisms are obtained by excision, and the remaining two maps are induced by inclusions, hence the diagram commutes. But notice that the dashed map is an isomorphism, by the first case, since $y \in \mathrm{int}(\Sigma)$. Therefore that top horizontal map is an isomorphism, as desired. □

1.2. The 2-dimensional cobordism category

Definition 1.6. The *(2-dimensional) cobordism category* $\mathcal{S}$ is defined as follows:

- objects of $\mathcal{S}$ are oriented closed 1-manifolds,
- given two objects M_0 and M_1, the set of morphisms $\mathcal{S}(M_0, M_1)$ is defined to be the set of equivalence classes of cobordisms from M_0 to M_1,
- composition of morphisms is defined on representatives, as described in section 1.1,
- the identity on M is represented by the cylinder $M \times I$, seen as a cobordism $M \to M$.

Observe that the composition in $\mathcal{S}$ is well defined. Indeed, suppose that we have cobordisms as follows:

$$\begin{tikzcd}
M_0 \arrow[r, shift left, "\Sigma_0"] \arrow[r, shift right, "\Sigma_1"'] & M_1 \arrow[r, shift left, "\Sigma'_0"] \arrow[r, shift right, "\Sigma'_1"'] & M_2,
\end{tikzcd} \tag{1.11}$$

and a pair of homeomorphisms $\Sigma_0 \cong \Sigma_1$ and $\Sigma'_0 \cong \Sigma'_1$ (relative to boundaries). Then the universal property of the pushout induces a homeomorphism $\Sigma'_0\Sigma_0 \cong \Sigma'_1\Sigma_1$, relative to the boundary. Thus $\Sigma_0 \sim \Sigma_1$ and $\Sigma'_0 \sim \Sigma'_1$ implies $\Sigma'_0\Sigma_0 \sim \Sigma'_1\Sigma_1$.

Morphisms in the cobordism category

One advantage of working in low dimensions is that complete classifications of manifolds are available, so that the objects and morphisms of $\mathcal{S}$ can be described precisely. For instance, the objects of $\mathcal{S}$ are exactly the spaces homeomorphic to a disjoint union of finitely many circles. In order to describe the morphisms, we recall some basic facts about the Euler characteristic of a manifold. A detailed account of the classification theorem for 2-manifolds is beyond the scope of this text, but we give pointers to the relevant literature.

It can be shown[1] that any compact manifold with (possibly empty) boundary is homeomorphic to a finite CW complex, so that the following definition makes sense for manifolds.

Definition 1.7. Let X be a finite CW complex. The *Euler characteristic* $\chi(X)$ of X is defined as the integer

$$\chi(X) = \sum_n (-1)^n c_n, \tag{1.12}$$

where c_n denotes the number of n-cells of X.

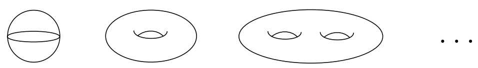
The Euler characteristic can be computed in terms of homology groups[2], hence it does not depend of a choice of CW structure. One immediate property of the Euler characteristic is as follows. Let X be a finite CW complex decomposing as a pair of subcomplexes, say $X = A \cup B$. Then $c_n(X) = c_n(A) + c_n(B) - c_n(A \cap B)$, and therefore the Euler characteristic is *subadditive* with respect to union:

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B). \tag{1.13}$$

In particular, the Euler characteristic is additive with respect to disjoint union:

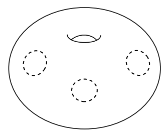
$$\chi(A \sqcup B) = \chi(A) + \chi(B). \tag{1.14}$$

Remark 1.8. By a *surface*, we mean a connected, compact 2-manifold. The classification theorem for surfaces [3] asserts that any closed orientable surface Σ is determined by its Euler characteristic $\chi(\Sigma)$, up to homeomorphism. Moreover, $\chi(\Sigma)$ is an even integer less than or equal to 2. The integer $1 - \frac{1}{2}\chi(\Sigma)$ is called the *genus* of the surface Σ. Thus, for every integer $g \geqslant 0$, there is a unique (up to homeomorphism) closed orientable surface of genus g, denoted Σ_g. For example, Σ_0 is the sphere, Σ_1 is the torus, etc.



Next, for fixed g, one can then obtain a surface with boundary by removing any number $k \geqslant 0$ of discs from Σ_g. The surface thus obtained is unique up to homeomorphism, and we denote it $\Sigma_{g,k}$. Here the number g is again called the *genus* of $\Sigma_{g,k}$. For instance, the surface $\Sigma_{1,3}$ can be depicted as follows:

[1] 1.
[2] 4, Theorem 2.44.
[3] [3] gives an overview.



By our definition of genus, we have $\chi(\Sigma_g) = 2 - 2g$. For surfaces with boundary, we see that the surface $\Sigma_{g,k-1}$ is obtained from the surface $\Sigma_{g,k}$ by attaching a disc D^1 along a circle S^1. Since clearly $\chi(S^1) = 0$ and $\chi(D^1) = 1$, eq. (1.13) shows inductively that

$$\chi(\Sigma_{g,k}) = 2 - 2g - k. \tag{1.15}$$

We note that the two formulas agree when $k = 0$.

Next we see how surfaces of the type Σ_g or $\Sigma_{g,k}$ define cobordisms, and thus morphisms in the cobordism category $\mathcal{S}$.

By remark 1.3, for a fixed choice of domain and target, a connected morphism in Σ is uniquely determined by its genus. When Σ is homeomorphic to $\Sigma_{g,k}$ for some g, k, then we shall say that Σ is of ***type*** $\Sigma_{g,k}$.

Symmetric monoidal structure

Let us equip the cobordism category with a symmetric monoidal structure. The first piece of datum required is that of a "tensor product" on $\mathcal{S}$, which is given by disjoint union:

$$\sqcup : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}. \tag{1.16}$$

Concretely, this sends a pair of objects M, M' in $\mathcal{S}$ to their disjoint union $M \sqcup M'$. For morphisms, we let $\sqcup$ be given on representatives. Taking different representatives yields a disjoint union which is homeomorphic relative to boundaries, so $\sqcup$ is well-defined. It is easy to check that $\sqcup$ is indeed a functor. Clearly, the empty manifold constitutes a unit with respect to disjoint union.

In order to construct the associator, unitors, and braiding, we use the following trick to convert homeomorphisms into cobordisms. Let M and M' be compact, oriented 1-manifolds, and suppose that $\varphi : M \cong M'$ is an orientation-preserving homeomorphism. Then, taking the mapping cylinder of φ defines a morphism $C_\varphi : M \to M'$ in $\mathcal{S}$. This is an isomorphism, with inverse given by the mapping cylinder of φ^{-1}. In this way, homeomorphisms between objects of $\mathcal{S}$ induce isomorphisms (in $\mathcal{S}$) between these objects.

Now, given objects M, M', M'' in $\mathcal{S}$, we choose the associator to be the isomorphism induced by the natural homeomorphism $M \sqcup (M' \sqcup M'') \cong (M \sqcup M') \sqcup M''$. Similarly, the braiding is induced by the homeomorphism $M \sqcup M' \cong M' \sqcup M$. The unitors are simply given by identities, since $M \sqcup \varnothing = M = \varnothing \sqcup M$ on the nose. It is not hard to verify that these isomorphisms are natural, and satisfy the required axioms for $\mathcal{S}$ to be a symmetric monoidal category.

A skeleton for $\mathcal{S}$

Let us denote by m the object of $\mathcal{S}$ made of m disjoint copies of the standard oriented circle, and let M be an object of $\mathcal{S}$ with $m > 0$ connected components.

By the classification of 1-manifolds, there exists an orientation-preserving homeomorphism $\varphi_i : S^1 \to M_i$ from the standard oriented circle to M_i.

Collecting the f_i together yields an orientation-preserving homeomorphism

$$\bigsqcup_i \varphi_i : \bigsqcup_i S^1 \to M. \tag{1.17}$$

Now, taking the mapping cylinder of this homeomorphism then gives an isomorphism in the category $\mathcal{S}$, between the object M and the disjoint union of m circles with standard orientation. The set of objects $\{ m \in \mathbb{N} \}$ therefore constitutes a skeleton for the cobordism category. From now on, we shall frequently identify $\mathcal{S}$ with this skeleton.

We remark that the skeleton can be given a symmetric monoidal structure, using the symmetric monoidal structure on $\mathcal{S}$. With our notation, we then have $m \sqcup n = m + n$.

The functors χ and Θ

It is clear that the Euler characteristic of the circle is zero. Since the Euler characteristic is additive with respect to disjoint union, we therefore have $\chi(M) = 0$ for any object M in $\mathcal{S}$. Let $\Sigma : M_0 \to M_1$ and $\Sigma' : M_1 \to M_2$ be morphisms in $\mathcal{S}$, and consider their composite $\Sigma'\Sigma : M_0 \to M_2$. Observe that we have $\Sigma'\Sigma = \Sigma' \sqcup \Sigma$ and $\Sigma' \cap \Sigma = M_1$, hence by subadditivity of the Euler characteristic, we have

$$\chi(\Sigma'\Sigma) = \chi(\Sigma) + \chi(\Sigma'). \tag{1.18}$$

In conclusion, the Euler characteristic defines a functor

$$\mathcal{S} \xrightarrow{\chi} \mathbb{Z}, \tag{1.19}$$

where $\mathbb{Z}$ denotes the group of integers, seen as a one-object category, with the natural symmetric monoidal structure [4]. The functor χ is then a symmetric monoidal functor, by additivity.

Another functor of major interest is

$$\mathcal{S} \xrightarrow{\Theta} \mathbb{Z}, \tag{1.20}$$

defined as follows. On objects, Θ is (necessarily) constant. Let $\Sigma : M \to N$ be a morphism in $\mathcal{S}$, and suppose that M has m connected components, and N has n connected components. Then we let

$$\Theta(\Sigma) = \frac{1}{2}(n - m - \chi(\Sigma)). \tag{1.21}$$

[4] example B.2

Using that χ is a symmetric monoidal functor, it is easy to check that Θ is a symmetric monoidal functor as well. In the case when $\Sigma : m \to n$ is of type $\Sigma_{g,k}$, the value $\Theta(\Sigma)$ takes a nicer form:

$$\begin{aligned}\Theta(\Sigma_{g,k}) &= \frac{1}{2}(n - m - \chi(\Sigma_{g,k})) \\ &= \frac{1}{2}(n - m - (2 - 2g - k)) \\ &= \frac{1}{2}(n - m - (2 - 2g - (n + m))) \\ &= g + n - 1.\end{aligned} \tag{1.22}$$

1.3. Analysis of subcategories

The goal of this section is to present some results of Tillmann [8], who has computed the classifying spaces of several subcategories of $\mathcal{S}$. Our exposition closely follows that of [8]. We will be interested in the following two subcategories:

- $\mathcal{S}_{>0}$: the subcategory of $\mathcal{S}$ containing all objects of $\mathcal{S}$ except 0. The morphisms in the category $\mathcal{S}_{>0}$ are those for which every connected component has nonempty incoming and outgoing boundaries.
- $\mathcal{S}_1$: the full subcategory of $\mathcal{S}_{>0}$, with the circle as unique object. All the morphisms of $\mathcal{S}_1$ are connected and have exactly one incoming and one outgoing boundary component.

For instance, the following morphisms are in $\mathcal{S}$, but not in $\mathcal{S}_{>0}$:

Proposition 1.9. There is an isomorphism of monoids $\mathcal{S}_1 \cong \mathbb{N}$, between $\mathcal{S}_1$ and the additive monoid of natural numbers. In particular, there is a homotopy equivalence $\mathbf{B}\mathcal{S}_1 \simeq S^1$.

Proof. The morphisms in the category $\mathcal{S}_1$ are precisely the connected surfaces with two boundary circles. Such surfaces are classified by their genus, and composition in the category $\mathcal{S}_1$ corresponds to addition of genus. Thus there is an isomorphism of monoids

$$\mathcal{S}_1 \cong \mathbb{N}. \tag{1.23}$$

Thus we have $\mathbf{B}\mathcal{S}_1 \simeq \mathbf{B}\mathbb{N} \simeq S^1$, where the second homotopy equivalence is obtained via [7, Corollary 11]. □

Our first use of the functor Θ is to study the category $\mathcal{S}_{>0}$. First, observe that the restriction of Θ to $\mathcal{S}_{>0}$ takes values in the natural numbers:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Theta"] & \mathbb{Z} \\ \mathcal{S}_{>0} \arrow[u, hook] \arrow[r, "\Theta"'] & \mathbb{N}. \arrow[u, hook] \end{tikzcd} \tag{1.24}$$

To see this, it suffices to consider connected morphisms, by the additivity of Θ. Let $\Sigma : m \to n$ be a connected morphism in $\mathcal{S}_{>0}$, say of type $\Sigma_{g,k}$. By definition of $\mathcal{S}_{>0}$, we must have $n \geqslant 1$. Then, using eq. (1.22), we have $\Theta(\Sigma_{g,k}) = g + n - 1 \geqslant 0$ as desired.

Theorem 1.10. [8, Theorem 4] The functor $\Theta : \mathcal{S}_{>0} \to \mathbb{N}$ has a right adjoint Φ, such that the inclusion $\mathcal{S}_1 \hookrightarrow \mathcal{S}_{>0}$ factors through the isomorphism $\mathcal{S}_1 \cong \mathbb{N}$, as in the following diagram:

$$\begin{tikzcd} \mathcal{S}_1 \arrow[rr, hook] \arrow[dr, "\cong"'] & & \mathcal{S}_{>0} \\ & \mathbb{N}. \arrow[ur, "\Phi"'] & \end{tikzcd} \tag{1.25}$$

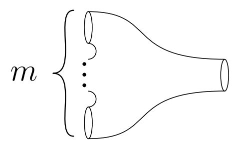
In particular, there is a homotopy equivalence $\mathbf{B}\mathcal{S}_{>0} \simeq S^1$.

Proof. Let $*$ denote the unique object of $\mathbb{N}$, and write 1 for the object of $\mathcal{S}_{>0}$ corresponding to the circle S^1. We define a functor Φ as follows:

$$\begin{aligned} \mathbb{N} &\xrightarrow{\quad \Phi \quad} \mathcal{S}_{>0} \\ * &\longmapsto 1 \\ (* \xrightarrow{m} *) &\longmapsto (1 \xrightarrow{\Sigma_{m,2}} 1), \end{aligned} \tag{1.26}$$

i.e. we let $\Phi(m)$ be the morphism $1 \to 1$ of genus m. We wish to show that that Φ is right adjoint to Θ.

We define natural transformations $\eta : 1_{\mathcal{S}_{>0}} \to \Phi\Theta$ and $\varepsilon : \Theta\Phi \to 1_{\mathbb{N}}$. Here ε is simply chosen to be the identity. For an object M of $\mathcal{S}_{>0}$ with m connected components, we let $\eta_M : M \to 1$ be the unique morphism of type $\Sigma_{0,m+1}$.



$= \eta_M$

Let us verify that η is natural. Let $\Sigma : M \to N$ be a morphism in $\mathcal{S}_{>0}$, with M having m components and N having n components. Naturality of η_M amounts to the commutativity of the following diagram:

$$
\begin{tikzcd}
M \arrow[r, "\Sigma"] \arrow[d, "\eta_M"'] & N \arrow[d, "\eta_N"] \\
1 \arrow[r, "\widetilde{\Sigma}"'] & 1,
\end{tikzcd}
\tag{1.27}
$$

where $\widetilde{\Sigma}$ denotes the morphism $\Phi\Theta(\Sigma) : 1 \to 1$ of genus $\Theta(\Sigma)$.

Now, for eq. (1.27) to commute, the two composites (↳) and (⤵) must be equal. Since both composites have the same source and target, it suffices to check that their Euler characteristics are equal.

Note that the surface $\widetilde{\Sigma}$ has two boundary components and has genus $\Theta(\Sigma)$, hence its Euler characteristic is as follows:

$$
\begin{aligned}
\chi(\widetilde{\Sigma}) &= 2 - 2\Theta(\Sigma) - 2 \\
&= 2 - 2\left(\frac{1}{2}\left(n - m - \chi(\Sigma)\right)\right) - 2 \quad \text{by definition of } \Theta \text{ (from eq. (1.21))} \\
&= \chi(\Sigma) + m - n.
\end{aligned}
\tag{1.28}
$$

Using that $\chi(\eta_M = 2 - (m+1))$, we now see that the two composites are equal:

$$
\begin{aligned}
\chi(↳) &= \chi(\widetilde{\Sigma}) + \chi(\eta_M) \\
&= (\chi(\Sigma) + m - n) + (2 - (m+1)) \\
&= \chi(\Sigma) + (2 - (n+1)) \\
&= (\chi(\Sigma) + \chi(\eta_N) \\
&= \chi(⤵).
\end{aligned}
\tag{1.29}
$$

Thus η is natural.

To obtain the adjunction $\Theta \dashv \Phi$, it remains to verify that η and ε satisfy the triangle identities. That is, we need the following pair of diagrams to commute:

$$
\begin{tikzcd}
\Theta(M) \arrow[r, "\Theta(\eta_M)"] \arrow[dr, "1_{\Theta_M}"'] & \Theta\Phi\Theta(M) \arrow[d, "\varepsilon_{\Theta(M)}"] \\
& \Theta(M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi(*) \arrow[r, "\eta_{\Phi(*)}"] \arrow[dr, "1_{\Phi(*)}"'] & \Phi\Theta\Phi(*) \arrow[d, "\Phi(\varepsilon_*)"] \\
& \Phi(*),
\end{tikzcd}
\tag{1.30}
$$

for any object M in $\mathcal{S}_{>0}$. But these commute trivially, since ε is the identity.

We conclude that Φ is left-adjoint to Θ, and therefore $\mathrm{B}\mathcal{S}_{>0} \simeq \mathrm{B}\mathbb{N} \simeq S^1$, where the last equivalence is obtained via [7, Corollary 11]. Finally, commutativity of eq. (1.25) is immediate. $\square$

2. The cobordism groupoid

Roughly speaking the *groupoid completion* $\mathcal{GC}$ of a small category $\mathcal{C}$ is a groupoid obtained from $\mathcal{C}$ by formally inverting morphisms. Section 2.1 formally defines and constructs this object, and shows that when $\mathcal{C}$ is symmetric monoidal, then $\mathcal{GC}$ is symmetric monoidal in a natural way. In section 2.2, we prove that there is a symmetric monoidal equivalence $\mathcal{GS} \simeq \mathbb{Z}$, a result due to Tillmann [8]. Our proof is a slight simplification of Tillmann's, and makes it obvious that under this equivalence, the generator of $\mathbb{Z}$ corresponds to the sphere.

2.1. Groupoid completion

We define the groupoid completion of a small category via a universal property, and show that it always exists. Moreover, taking groupoid completion is functorial, and turns out to preserve finite products. As a result, the groupoid completion of a symmetric monoidal category is itself symmetric monoidal. With this structure, the groupoid completion enjoys a universal property in the context of symmetric monoidal categories.

Definition and construction

Definition 2.1. A *groupoid completion* of a category $\mathcal{C}$ consists of a groupoid $\mathcal{GC}$ and a functor $q_{\mathcal{C}} : \mathcal{C} \to \mathcal{GC}$, such that any functor $\mathcal{C} \to \mathcal{E}$ to a groupoid $\mathcal{E}$ factors uniquely through $q_{\mathcal{C}}$, as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\forall"] \arrow[d, "q_{\mathcal{C}}"'] & \mathcal{E} \\ \mathcal{GC}. \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \tag{2.1}$$

When a groupoid completion exists, it is unique up to unique isomorphism, and we speak of *the* groupoid completion of the category. Roughly, the groupoid $\mathcal{GC}$ is obtained by adjoining formal inverses to the category $\mathcal{C}$. Formally, this is done by a free construction followed by a quotient construction.

Proposition 2.2. Let $\mathcal{C}$ be a small category. Then a groupoid completion $\mathcal{GC}$ of $\mathcal{C}$ exists.

Proof. First, consider the oriented graph C constructed from the category $\mathcal{C}$ as follows: let the set of vertices of C be the set of objects of $\mathcal{C}$, and for any two vertices a and b, we let the set of edges from a to b be the disjoint union $\mathcal{C}(a,b) \sqcup \mathcal{C}^{\mathrm{op}}(a,b)$. The idea is that the morphism f^{op} in $\mathcal{C}^{\mathrm{op}}$ represents the inverse to a morphism f in $\mathcal{C}$. Let $\mathcal{P}$ be the free category on the graph C. Concretely, the objects of $\mathcal{P}$ are vertices (i.e. objects in $\mathcal{C}$), and the morphisms are finite strings $\langle l_0, l_1, l_2, \dots, l_n\rangle$ of composable edges of C. Here we say that two edges $l : a \to b$ and $l' : a' \to b'$ are composable whenever there is equality between any two objects in the set $\{\, a, b, a', b' \,\}$. Composition in $\mathcal{P}$ is given by concatenation of strings, and the identity morphism of an object a is represented by an empty string denoted $\langle\,\rangle_a$.

Second, we consider the equivalence relation $\sim$ on morphisms of $\mathcal{P}$ generated by the following:

- For all objects a in $\mathcal{C}$, $\langle\,\rangle_a \sim \langle 1_a\rangle$,
- for all composable morphisms f, g in $\mathcal{C}$, $\langle f, g\rangle \sim \langle g \circ f\rangle$,
- for every morphism f in $\mathcal{C}$, $\langle f, f^{\mathrm{op}}\rangle \sim \left\langle 1_{\mathrm{dom}(f)}\right\rangle$ and $\langle f^{\mathrm{op}}, f\rangle \sim \left\langle 1_{\mathrm{cod}(f)}\right\rangle$.

Now we define $\mathcal{GC}$ to be the quotient category $\mathcal{P}/\sim$. This is indeed a groupoid: the inverse to a string $\langle l_0, \dots, l_n\rangle$ is given by $\langle l_n^{\mathrm{op}}, \dots, l_0^{\mathrm{op}}\rangle$. Here l_i^{op} denotes the edge l_i with reverse orientation, which is an edge in $\mathcal{P}$ by construction.

There is a canonical functor $q_{\mathcal{C}} : \mathcal{C} \to \mathcal{GC}$ which is the identity on objects and sends a morphism f to the equivalence class of $\langle f\rangle$. Lastly, for the universal property, we see that any functor $\mathcal{C} \to \mathcal{E}$ to a groupoid $\mathcal{E}$ uniquely determines a functor $\mathcal{P} \to \mathcal{E}$. This functor respects the elementary equivalences above, hence it uniquely factors through the quotient category $\mathcal{P}/\sim$. $\square$

A string $\left\langle f_0, f_1^{\mathrm{op}}, f_2, \dots, f_{n-1}^{\mathrm{op}}, f_n\right\rangle$ from the above construction is commonly represented by a zig-zag of morphisms

$$\begin{tikzcd}
& \bullet & & \cdots & & \bullet. \\
\bullet \arrow[ur, "f_0"] & & \bullet \arrow[ul, "f_1"] \arrow[ur, "f_2"] & & \bullet \arrow[ul, "f_{n-1}"] \arrow[ur, "f_n"] &
\end{tikzcd} \tag{2.2}$$

In this representation, formal inverses are shown pointing from right to left. By composing morphisms in $\mathcal{C}$ and inserting identities as needed, any element of $\mathcal{GC}$ can be given a representative of the above form. We think of the above string as a composite with formal inverses, $f_n \circ f_{n-1}^{-1} \circ \dots \circ f_2 \circ f_1^{-1} \circ f_0$.

One example of interest is when the category $\mathcal{C}$ is a monoid, i.e. a category with only one object.

Example 2.3. The groupoid completion $\mathcal{G}M$ of a monoid M is a groupoid with one object, i.e. a group. By construction, the group $\mathcal{G}M$ admits the following presentation:

generators are elements of M, and relations are defined by the multiplication table of M. We now focus on a particular case. Let x be an object in a category $\mathcal{C}$, and consider the monoid $M = \mathrm{End}_{\mathcal{C}}(x)$. A priori, there are two competing ways to obtain a group from M. On the one hand, we have the group completion $\mathcal{G}M$. On the other hand, the object x is also an object of the groupoid $\mathcal{GC}$, and thus we have the group $G = \mathrm{Aut}_{\mathcal{GC}}(x)$. The groups $\mathcal{G}M$ and G are related as follows. The functor $q_{\mathcal{C}}$ restricts to a functor $q_{\mathcal{C}} : M \to G$, which by the universal property of $\mathcal{G}M$ induces a unique functor p making the following diagram commute:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "q_{\mathcal{C}}"] & & \mathcal{GC} \\ M \arrow[u, hook] \arrow[rr, "q_{\mathcal{C}}"] \arrow[dr, "q_M"'] & & G \arrow[u, hook] \\ & \mathcal{G}M. \arrow[ur, dashed, "p"'] & \end{tikzcd} \tag{2.3}$$

We introduce a condition on the category $\mathcal{C}$ which makes the map p constructed in example 2.3 surjective.

Definition 2.4. A category $\mathcal{C}$ is said to be ***strongly connected*** at an object x of $\mathcal{C}$ if for any other object y in $\mathcal{C}$, there exist morphisms $x \to y$ and $y \to x$ in $\mathcal{C}$.

Lemma 2.5. Let $\mathcal{C}$ be a category strongly connected at x. Let $M = \mathrm{End}_{\mathcal{C}}(x)$ and $G = \mathrm{Aut}_{\mathcal{GC}}(x)$. Then the canonical map $p : \mathcal{G}M \to G$ is surjective.

Proof. Any automorphism $f : x \to x$ in $\mathcal{GC}$ is represented by a zigzag of morphisms in $\mathcal{C}$, say:

$$x \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \dots \longleftarrow \bullet \longrightarrow x. \tag{2.4}$$

Since $\mathcal{C}$ is strongly connected at x, there are morphisms between each $\bullet$ and x (represented by vertical arrows in the diagram below):

$$\begin{tikzcd} x \arrow[r] \arrow[dr, dashed, "f_0"'] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] & \dots & \bullet \arrow[l] \arrow[r] & x \\ & x & x \arrow[u] \arrow[l, dashed, "f_1"] & & x. \arrow[u] \arrow[ur, dashed, "f_n"'] & \end{tikzcd} \tag{2.5}$$

Here the dashed arrows simply represent composite morphisms. Then f can be represented by the equivalent zigzag

$$x \xrightarrow{f_0} x \xleftarrow[f_1]{} x \xrightarrow{f_2} \dots \xleftarrow[f_{n-1}]{} x \xrightarrow{f_n} x, \tag{2.6}$$

that is, by an element of $\mathcal{G}\,\mathrm{End}_C(x)$, since each f_i is an endomorphism of x. This shows that the map $\mathcal{G}M \to G$ is surjective. $\square$

Groupoid completion as a functor on Cat

Given a functor $F : \mathcal{C} \to \mathcal{C}'$ between small categories, the universal property of the groupoid completion induces a functor $\mathcal{G}F : \mathcal{GC} \to \mathcal{GC}'$ as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q_{\mathcal{C}}"'] & \mathcal{C}' \arrow[d, "q_{\mathcal{C}'}"] \\ \mathcal{GC} \arrow[r, dashed, "\mathcal{G}F"'] & \mathcal{GC}'. \end{tikzcd} \tag{2.7}$$

It is easy to check that this defines a functor $\mathcal{G}(-) : \mathbf{Cat} \to \mathbf{Gpd}$. This functor turns out to be a left-adjoint. Indeed, the universal property of the groupoid completion can be stated as a natural bijection of sets: for any small category $\mathcal{D}$ and any small groupoid $\mathcal{E}$, we have

$$\mathbf{Gpd}(\mathcal{GD}, \mathcal{E}) \cong \mathbf{Cat}(\mathcal{D}, \mathcal{E}), \tag{2.8}$$

Here the sets of functors can be given the structure of categories, with natural transformations as morphisms (see example A.3 for details), and the above bijection can be enriched:

Lemma 2.6. For any groupoid $\mathcal{E}$ there is an isomorphism of categories

$$\underline{\mathbf{Gpd}}(\mathcal{GD}, \mathcal{E}) \cong \underline{\mathbf{Cat}}(\mathcal{D}, \mathcal{E}). \tag{2.9}$$

Proof. We show that both maps participating in the bijection from eq. (2.8) can be made into functors. Going from left to right is done by precomposing with $q_{\mathcal{D}}$, and this is clearly functorial. Let $\tau : F \to G$ be a natural transformation between functors $\mathcal{D} \to \mathcal{E}$. We define a natural transformation $\hat{\tau} : \hat{F} \to \hat{G}$ between the functors induced by the universal property by letting $\hat{\tau}_c = \tau_c$ for every object c in $\mathcal{D}$. This makes sense because $\mathcal{GD}$ has the same objects as $\mathcal{D}$. Naturality of $\hat{\tau}$ follows from the naturality of τ, since it is enough to check naturality with respect to morphisms of the form $q_{\mathcal{D}}(f)$ for f a morphism in $\mathcal{D}$. The assignment $\tau \mapsto \hat{\tau}$ clearly respects identities and composition, so we indeed have a functor from right to left. □

The bijection from eq. (2.8) basically states that the functor $\mathcal{G}(-)$ is a left-adjoint to the inclusion of **Gpd** into **Cat**.

$$\begin{tikzcd} \mathbf{Cat} \arrow[rr, bend left, "\mathcal{G}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Gpd}. \arrow[ll, hook, bend left] \end{tikzcd} \tag{2.10}$$

By general nonsense, the functor $\mathcal{G}(-)$ therefore preserves arbitrary colimits. It turns out that it also preserves finite products. Our proof of this fact relies on the *exponential* *law* for categories, which we briefly recall. Given small categories $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$, there is a natural bijection

$$\mathbf{Cat}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathbf{Cat}(\mathcal{X}, \underline{\mathbf{Cat}}(\mathcal{Y}, \mathcal{Z})). \tag{2.11}$$

Succintly: a functor $(x, y) \mapsto F(x, y)$ is sent to the functor $x \mapsto (y \mapsto F(x, y))$. If $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are small groupoids, then $\underline{\mathbf{Gpd}}(\mathcal{Y}, \mathcal{Z})$ is a groupoid. Indeed, a natural transwormation τ between functors $\mathcal{Y} \to \mathcal{Z}$ is automatically an isomorphism, since each component τ_y is an isomorphism. Thus we have a version of the exponential law for groupoids:

$$\mathbf{Gpd}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathbf{Gpd}(\mathcal{X}, \underline{\mathbf{Gpd}}(\mathcal{Y}, \mathcal{Z})). \tag{2.12}$$

The proof of the following result is now purely formal.

Lemma 2.7. For any pair of small categories $\mathcal{C}$ and $\mathcal{D}$, there is a natural isomorphism $\mathcal{G}(\mathcal{C} \times \mathcal{D}) \cong \mathcal{G}\mathcal{C} \times \mathcal{G}\mathcal{D}$.

Proof. By the Yoneda lemma, it suffices to exhibit a natural isomorphism between the functors represented by $\mathcal{G}(\mathcal{C} \times \mathcal{D})$ and $\mathcal{G}\mathcal{C} \times \mathcal{G}\mathcal{D}$. For any groupoid $\mathcal{E}$, we have natural isomorphisms

$$\begin{aligned}
\mathbf{Gpd}(\mathcal{G}\mathcal{C} \times \mathcal{G}\mathcal{D}, \mathcal{E}) &\cong \mathbf{Gpd}(\mathcal{G}\mathcal{C}, \underline{\mathbf{Gpd}}(\mathcal{G}\mathcal{D}, \mathcal{E})) && \text{by the exponential law for groupoids} \\
&\cong \mathbf{Cat}(\mathcal{C}, \underline{\mathbf{Gpd}}(\mathcal{G}\mathcal{D}, \mathcal{E})) && \text{by the uni. prop. of } \mathcal{G}\mathcal{C} \\
&\cong \mathbf{Cat}(\mathcal{C}, \underline{\mathbf{Cat}}(\mathcal{D}, \mathcal{E})) && \text{by eq. (2.9)} \\
&\cong \mathbf{Cat}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) && \text{by the exponential law for categories} \\
&\cong \mathbf{Gpd}(\mathcal{G}(\mathcal{C} \times \mathcal{D}), \mathcal{E}) && \text{by the uni. prop. of } \mathcal{G}(\mathcal{C} \times \mathcal{D}).
\end{aligned}$$

□

Remark 2.8. While the isomorphism of lemma 2.7 was obtained using abstract machinery, tracing through the definitions reveals that it simply makes morphisms of the form $q_{\mathcal{C}\times\mathcal{D}}(f, g)$ correspond to morphisms of the form $(q_{\mathcal{C}}(f), q_{\mathcal{D}}(g))$.

Symmetric monoidal structure

Lemma 2.7 makes the following definition possible. Given a symmetric monoidal category $\mathcal{C}$ with tensor product $\otimes$, we define a functor $\otimes'$ as follows:

$$\begin{tikzcd} \mathcal{G}\mathcal{C} \times \mathcal{G}\mathcal{C} \arrow[r, "\cong"] \arrow[rr, bend right, "\otimes'"'] & \mathcal{G}(\mathcal{C} \times \mathcal{C}) \arrow[r, "\mathcal{G}(\otimes)"] & \mathcal{G}\mathcal{C}. \end{tikzcd} \tag{2.13}$$

To show that this functor is part of a symmetric monoidal structure on $\mathcal{G}\mathcal{C}$, we begin with an observation.

Remark 2.9. Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[rr, "\otimes"] \arrow[d, "q_{\mathcal{C}\times\mathcal{C}}"'] & & \mathcal{C} \arrow[dd, "q_{\mathcal{C}}"] \\ \mathcal{G}(\mathcal{C} \times \mathcal{C}) \arrow[d, "\cong"'] \arrow[drr, "\mathcal{G}(\otimes)"] & & \\ \mathcal{G}\mathcal{C} \times \mathcal{G}\mathcal{C} \arrow[rr, "\otimes'"'] & & \mathcal{G}\mathcal{C}. \end{tikzcd} \tag{2.14}$$

Here the square commutes by definition of $\mathcal{G}(\otimes)$, and the triangle commutes by definition of $\otimes'$. Using the explicit description of the isomorphism (given in remark 2.8), we see that commutativity of the above diagram indicates that we have

$$q_{\mathcal{C}}(f \otimes g) = q_{\mathcal{C}}(f) \otimes' q_{\mathcal{C}}(g), \tag{2.15}$$

for any pair of morphisms f and g in $\mathcal{C}$.

Proposition 2.10. The groupoid completion of a symmetric monoidal category is a symmetric monoidal category.

Proof. Let $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ be a symmetric monoidal category. The symmetric monoidal structure on the groupoid completion $\mathcal{G}\mathcal{C}$ is given by $(\otimes', e, q_{\mathcal{C}}(\alpha), q_{\mathcal{C}}(\lambda), q_{\mathcal{C}}(\rho), q_{\mathcal{C}}(\gamma))$, where the tensor product $\otimes'$ is defined as in eq. (2.13).

The fact that this indeed defines a symmetric monoidal structure follows from remark 2.9. To illustrate, we verify the triangle axiom. Let a and c be objects in $\mathcal{G}\mathcal{C}$ (i.e. objects in $\mathcal{C}$). The triangle axiom for $\mathcal{C}$ states that the following diagram commutes.

$$\begin{tikzcd} a \otimes (e \otimes c) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes\lambda"'] & & (a \otimes e) \otimes c \arrow[dl, "\rho\otimes 1"] \\ & a \otimes c. & \end{tikzcd} \tag{2.16}$$

Applying the functor $q_{\mathcal{C}}$, we see that the following diagram commutes.

$$\begin{tikzcd} a \otimes (e \otimes c) \arrow[rr, "q_{\mathcal{C}}(\alpha)"] \arrow[dr, "1\otimes q_{\mathcal{C}}(\lambda)"'] & & (a \otimes e) \otimes c \arrow[dl, "q_{\mathcal{C}}(\rho)\otimes 1"] \\ & a \otimes c. & \end{tikzcd} \tag{2.17}$$

Here we have used that $q_{\mathcal{C}}(1 \otimes \lambda) = 1 \otimes q_{\mathcal{C}}(\lambda)$ and $q_{\mathcal{C}}(\rho \otimes 1) = q_{\mathcal{C}}(\rho) \otimes 1$, which follows from remark 2.9 and functoriality of $q_{\mathcal{C}}$. This verifies the triangle axiom for $\mathcal{G}\mathcal{C}$, and the other axioms are verified similarly. $\square$

Finally, we provide an analog of eq. (2.9) in the context of symmetric monoidal categories.

Proposition 2.11. Let $\mathcal{C}$ and $\mathcal{C}'$ be small symmetric monoidal categories, and assume that $\mathcal{C}'$ is a groupoid. Then there is an isomorphism of categories $\underline{\mathrm{SM}}(\mathcal{GC}, \mathcal{C}') \cong \underline{\mathrm{SM}}(\mathcal{C}, \mathcal{C}')$.

Proof. The functor $q_{\mathcal{C}}$ is naturally a symmetric monoidal functor $(q_{\mathcal{C}}, q_2, q_0)$, by choosing q_2 and q_0 to be identities. This makes sense because the objects of $\mathcal{GC}$ are just objects of $\mathcal{C}$. Since SM is a 2-category, precomposition by $q_{\mathcal{C}}$ defines a functor $\underline{\mathrm{SM}}(\mathcal{GC}, \mathcal{C}') \to \underline{\mathrm{SM}}(\mathcal{C}, \mathcal{C}')$.

We define a functor in the other direction, first on objects, and then on morphisms.

Let (F, F_2, F_0) be a symmetric monoidal functor $\mathcal{C} \to \mathcal{C}'$. By the universal property, F induces a functor $\hat{F} : \mathcal{GC} \to \mathcal{C}'$, since $\mathcal{C}'$ is a groupoid. We make $\hat{F}$ into a symmetric monoidal functor by letting $\hat{F}_0 = F_0$ and $\hat{F}_2 = F_2$. This makes sense, because $\mathcal{GC}$ and $\mathcal{C}$ have the same objects. Naturality of $\hat{F}_2$ follows from the naturality of F_2. Indeed, it suffices to check naturality with respect to morphisms in the image of $q_{\mathcal{C}}$. It is easy to check that $(\hat{F}, \hat{F}_2, \hat{F}_0)$ is a symmetric monoidal functor.

Next, let $\tau : F \to G$ be a morphism in $\underline{\mathrm{SM}}(\mathcal{C}, \mathcal{C}')$. We define a natural transformation $\hat{\tau} : \hat{F} \to \hat{G}$ by letting $\hat{\tau}_c = \tau_c$ for any object c in $\mathcal{C}$. Again, this makes sense because $\mathcal{GC}$ and $\mathcal{C}$ have the same objects, and it is easy to see that $\hat{\tau}$ is a symmetric monoidal natural transformation.

It is clear that the assignment $F \mapsto \hat{F}$ is a functor, and the universal property of $\mathcal{GC}$ shows that it is an inverse to the functor defined by precomposition by $q_{\mathcal{C}}$.

□

2.2. The cobordism groupoid

Our next goal is to prove the following result:

Theorem 2.12 ([8, Theorem 7]). There is a symmetric monoidal equivalence $\mathcal{GS} \simeq \mathbb{Z}$ between the groupoid completion of the cobordism category and the group of integers.

The symmetric monoidal structure on $\mathcal{GS}$ is inherited from that of $\mathcal{S}$, as explained in section 2.1. Here the group $\mathbb{Z}$ is seen as a one-object category, whose (strict) symmetric monoidal structure is given by addition of integers. We split the proof of theorem 2.12 into two results: lemma 2.13, which is formal, and theorem 2.15, which has geometric content.

Lemma 2.13. Let $\mathcal{C}$ be a small symmetric monoidal category. Let $G_e = \mathrm{Aut}_{\mathcal{GC}}(e)$ be the group of automorphisms of the unit object e, seen as the full subcategory of $\mathcal{GC}$, with e as its only object. If $\mathcal{C}$ is connected, then the inclusion $G_e \hookrightarrow \mathcal{GC}$ induces a symmetric monoidal equivalence

$$G_e \simeq \mathcal{GC}. \tag{2.18}$$

Proof. According to [2, 2.4.10], it suffices to show that the inclusion $G_e \hookrightarrow \mathcal{GC}$ is a symmetric monoidal functor and an equivalence of categories.

The Eckmann-Hilton argument shows that the monoid G_e is abelian (see appendix B.1). Therefore, G_e is a symmetric monoidal category in a natural way, with tensor product given by composition/multiplication. In example B.6, we show that the inclusion $G_e \hookrightarrow \mathcal{GC}$ is naturally a symmetric monoidal functor.

If $\mathcal{C}$ is connected, then so is $\mathcal{GC}$, and all objects in $\mathcal{GC}$ are isomorphic. Thus the the inclusion $G_e \hookrightarrow \mathcal{GC}$ is fully faithful and essentially surjective, i.e. an equivalence of categories. □

Since the cobordism category $\mathcal{S}$ is clearly connected, the above result reduces the study of $\mathcal{GS}$ to that of $G_0 = \mathrm{Aut}_{\mathcal{GS}}(0)$. The group G_0 can be related to the monoid $M_0 = \mathrm{End}_{\mathcal{S}}(0)$ by restricting the groupoid completion functor $q_{\mathcal{S}}$, as in the following commutative diagram:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "q_{\mathcal{S}}"] & \mathcal{GS} \\ M_0 \arrow[u, hook] \arrow[r, "q"'] & G_0. \arrow[u, hook] \end{tikzcd} \tag{2.19}$$

This leads us to take a closer look at the monoid M_0.

Remark 2.14. There is an isomorphism of monoids $M_0 \cong \mathbb{N}^{(\mathbb{N})}$. Here $\mathbb{N}^{(\mathbb{N})}$ denotes the abelian monoid of non-negative integer sequences with finite support (with pointwise addition). To see this, note that any morphism $0 \to 0$ in $\mathcal{S}$ is a disjoint union of finitely many closed surfaces, each of which is determined by its genus. In this way, the sequence $\omega = (\omega_0, \omega_1, \ldots)$ represents the morphism $0 \to 0$ consisting of ω_0 spheres, ω_1 tori, ω_2 double tori, and so on. It is clear that this is a bijection and a monoid homomorphism.

The monoid $\mathbb{N}^{(\mathbb{N})}$ is generated by elements of the form e_i for $i \geqslant 0$. Here e_j is the sequence which is identically 0, except in the i^{th} entry, where it is 1. Under the above isomorphism, the sequence e_i element corresponds to a the closed surface of genus i.

It is easy to check that the the group of integer sequences with finite support, $\mathbb{Z}^{(\mathbb{N})}$, satisfies the universal property of the group completion of $\mathbb{N}^{(\mathbb{N})}$, so that we have $\mathcal{G}M_0 \cong \mathbb{Z}^{(\mathbb{N})}$.

We now state the theorem which consitutes the highlight of Tillmann's paper [8]. We write S^2 for the morphism $1 \to 1$ in $\mathcal{S}$ given by the sphere, and we write $[S^2]$ for its image under the canonical functor $q_{\mathcal{S}} : \mathcal{S} \to \mathcal{GS}$.

Theorem 2.15. [8, Theorem 7] In particular, there is an isomorphism

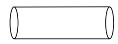
$$\mathrm{Aut}_{\mathcal{GS}}(0) \cong \mathbb{Z}, \tag{2.20}$$

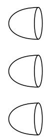
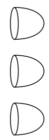
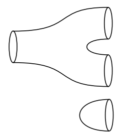
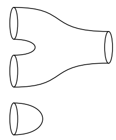
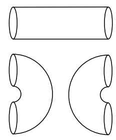
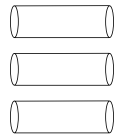
where the generator is given by $[S^2]$.

Proof. Let us first establish some convenient notation. For any morphism ω in $\mathcal{S}$, the image $q_{\mathcal{S}}(\omega)$ is denoted $[\omega]$. For any object x in $\mathcal{S}$, the monoid of endomorphisms $\mathrm{End}_{\mathcal{S}}(x)$ is denoted M_x, and the group of automorphisms $\mathrm{Aut}_{\mathcal{GS}}$ is denoted G_x. In order to facilitate the translation between symbolic notation and the associated pictures, we adopt the symbol "$\cdot$" to denote composition of morphisms in reverse order. That is, given any composable morphisms φ and ψ, the composite $\varphi \circ \psi$ is denoted $\psi \cdot \varphi$, as in the following diagram:

$$\begin{tikzcd} \bullet \arrow[r, "\psi"'] \arrow[rr, bend left, "\psi\cdot\varphi"] & \bullet \arrow[r, "\varphi"'] & \bullet. \end{tikzcd} \tag{2.21}$$

We now begin the proof. Consider the following morphisms in $\mathcal{S}$:

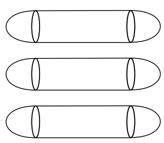
$\alpha : 0 \to 1$	$\beta : 1 \to 0$	$\gamma : 1 \to 1$
		

$\delta : 0 \to 3$	$\epsilon : 3 \to 0$	$\zeta : 1 \to 3$	$\eta : 3 \to 1$	$\theta : 3 \to 3$	$\iota : 3 \to 3$
					

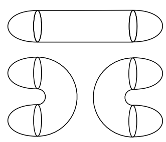
Consider the set map $f : G_3 \to G_0$ obtained by precomposing by $[\delta]$ and postcomposing by $[\varepsilon]$:

$$f : G_3 \xrightarrow{[\delta]\cdot(-)\cdot[\varepsilon]} G_0. \tag{2.22}$$

Observe that $\delta \cdot \iota \cdot \varepsilon = \delta \cdot \theta \cdot \varepsilon$ in M_0:



$$=$$



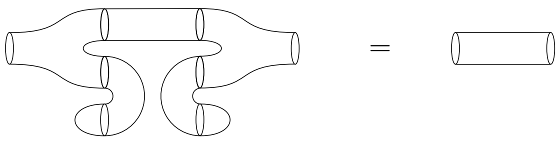
Therefore we have

$$\begin{aligned} f([\iota]) &= [\delta] \cdot [\iota] \cdot [\varepsilon] \\ &= [\delta \cdot \iota \cdot \varepsilon] \\ &= [\delta \cdot \theta \cdot \varepsilon] \\ &= [\delta] \cdot [\theta] \cdot [\varepsilon] \\ &= f([\theta]). \end{aligned} \tag{2.23}$$

The map f is a bijection, since $[\delta]$ and $[\varepsilon]$ are isomorphisms, so we conclude that

$$[\iota] = [\theta]\,. \tag{2.24}$$

Next, observe that $\zeta \cdot \theta \cdot \eta = \gamma$ in M_1:



Thus we have

$$\begin{aligned}[\zeta \cdot \eta] &= [\zeta \ \ \cdot \iota \cdot \eta] \\ &= [\zeta] \cdot [\iota] \cdot [\eta] \\ &= [\zeta] \cdot [\theta] \cdot [\eta] \quad \text{by eq. (2.24)} \\ &= [\zeta \cdot \theta \cdot \eta] \\ &= [\gamma]\end{aligned} \tag{2.25}$$

And for any integer $i \geqslant 0$, we have

$$\begin{aligned}[e_i + ie_0] &= \left[\alpha \cdot (\zeta \cdot \eta)^i \cdot \beta\right] \\ &= [\alpha] \cdot [\zeta \cdot \eta]^i \cdot [\alpha] \\ &= [\alpha] \cdot [\gamma]^i \, [\beta] && \text{by eq. (2.25)} \\ &= [\alpha] \cdot [\beta] && \text{since } [\gamma] \text{ is the identity in } G_1 \\ &= [\alpha \cdot \beta] \\ &= [e_0]\,.\end{aligned} \tag{2.26}$$

In the first and last equalities, we identify M_0 with $\mathbb{N}^{(\mathbb{N})}$, as in remark 2.14. Since $[-]$ is a group homomorphism, and G_0 is an abelian group (see appendix B.1), we can rewrite the above as

$$[e_i] = (1-i)\,[e_0]\,. \tag{2.27}$$

Since $[-]$ is a group homomorphism, and G_0 is an abelian group (see appendix B.1), we can rewrite the above as $[e_i] = (1-i)\,[e_0]$.

The Euler characteristic of the closed surface of genus i is $\chi(e_i) = -2(1-i)$. Thus we have shown that $[e_i] = -\frac{1}{2}\chi(e_i)\,[e_0]$ for $i \geqslant 0$. Since $[-]$ and χ are monoid homomorphisms, it follows that for any $\omega \in M_0$, we have

$$[\omega] = -\frac{1}{2}\chi(\omega)\,[e_0]\,. \tag{2.28}$$

Recall that the Euler characteristic defines a functor $\chi : \mathcal{S} \to \mathbb{Z}$ to the group of integers. By the universal property of the groupoid completion, there is an induced functor $\overline{\chi} : \mathcal{GS} \to \mathbb{Z}$. Thus we have a commutative diagram:

$$\begin{tikzcd}
M_0 \arrow[r, hook] \arrow[d, "q"'] & \mathcal{S} \arrow[d, "q_{\mathcal{S}}"] \arrow[dr, "\chi"] & \\
 & & \mathbb{Z} \\
G_0 \arrow[r, hook] & \mathcal{GS}, \arrow[ur, "\overline{\chi}"'] &
\end{tikzcd} \tag{2.29}$$

where the restriction of $\overline{\chi}$ to G_0 is a group homomorphism.

We wish to show that $\overline{\chi}$ is injective, using eq. (2.28). By lemma 2.5, there is a surjective group homomorphism $\mathcal{G}M_0 \to G_0$. Since the group $\mathcal{G}M_0$ is generated by elements of the monoid M_0, and G_0 is abelian, any element $\kappa \in G_0$ is of the form $\kappa = [\lambda] - [\mu]$, for some λ, μ in M_0. Then we have

$$\begin{aligned}
\kappa &= [\lambda] - [\mu] \\
&= -\frac{1}{2}\chi(\lambda)\,[e_0] - \left(-\frac{1}{2}\chi(\mu)\,[e_0]\right) \\
&= -\frac{1}{2}(\chi(\lambda) - \chi(\mu))\,[e_0]\,.
\end{aligned} \tag{2.30}$$

Now, if $\overline{\chi}(\kappa) = 0$, then $\overline{\chi}([\lambda]) - \overline{\chi}([\mu]) = 0$, i.e. $\chi(\lambda) - \chi(\mu) = 0$, by the commutativity of eq. (2.29). But then $\kappa = 0$, by eq. (2.30). We conclude that $\overline{\chi}$ is an injective group homomorphism.

Finally, since $\overline{\chi}([e_0]) = \chi(e_0) = 2$, the image of $\overline{\chi}$ is a nontrivial subgroup of $\mathbb{Z}$, therefore G_0 is isomorphic to the group of integers $\mathbb{Z}$. □

2.3. The homotopy type of the 2-dimensional cobordism category

A result of Quillen [6, Proposition 1] shows that for any small category $\mathcal{C}$, there is an isomorphism of groups $\pi_1(\mathrm{B}\mathcal{C}) \cong \pi_1(\mathrm{B}\mathcal{GC})$. Consequently, by the results of the previous section, we have ([8, Theorem 7])

$$\pi_1(\mathrm{B}\mathcal{S}) \cong \mathbb{Z}. \tag{2.31}$$

As an application of the results of the previous section, we present the following result due to Tillmann.

Theorem 2.16. [8, Theorem 10]. There exists a simply-connected infinite loop space X such that $\mathrm{B}\mathcal{S} \simeq X \times S^1$.

Proof. Recall from Section 1.3 that there is a functor Θ as follows.

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Theta"] & \mathbb{Z} \\ \mathcal{S}_{>0} \arrow[u, hook, "i"] \arrow[r, "\Theta"'] & \mathbb{N}. \arrow[u, hook, "j"'] \end{tikzcd} \tag{2.32}$$

Then by theorem 1.10, we have

$$\begin{tikzcd} \mathbf{B}\mathcal{S} \arrow[r, "\mathbf{B}\Theta"] & \mathbf{B}\mathbb{Z} \simeq S^1 \\ S^1 \simeq \mathbf{B}\mathcal{S}_{>0}. \arrow[u, hook, "\mathbf{B}i"] \arrow[ur, "\mathbf{B}(j\Theta)"'] & \end{tikzcd} \tag{2.33}$$

Since $\mathbf{B}(j\Theta)$ is a homotopy equivalence, there exists some map $t : \mathbf{B}\mathbb{Z} \to \mathbf{B}\mathcal{S}_{>0}$ such that $B(j\Theta)t \simeq 1_{\mathbf{B}\mathbb{Z}}$, hence we have

$$\mathbf{B}(\Theta)\mathbf{B}(i)t = \mathbf{B}(\Theta i)t = \mathbf{B}(j\Theta)t \simeq 1_{S^1}, \tag{2.34}$$

i.e. the map $s = \mathbf{B}(i)t$ is a homotopy section for $\mathbf{B}(\Theta)$.

Let X be the homotopy fiber of $\mathbf{B}(i) : S^1 \to \mathbf{B}\mathcal{S}$. Then the fiber sequence

$$X \xrightarrow{f} \mathbf{B}\mathcal{S} \xrightarrow{\Theta} S^1 \tag{2.35}$$

induces a long exact sequence on homotopy groups:

$$\ldots \pi_n X \xrightarrow{f_*} \pi_n \mathbf{B}\mathcal{S} \xrightarrow{\Theta_*} \pi_n S^1 \xrightarrow{\partial} \pi_{n-1} X \ldots \tag{2.36}$$

Since the map Θ_* has section s_*, it is surjective. Therefore ∂ is the zero map, and f_* is injective, so that the above long exact sequence yields short exact sequences

$$0 \longrightarrow \pi_n X \xrightarrow{f_*} \pi_n \mathbf{B}\mathcal{S} \xrightarrow{\Theta_*} \pi_n S^1 \longrightarrow 0, \tag{2.37}$$

for all $n \geqslant 0$. Using that the category $\mathcal{S}$ is connected, taking $n = 0$ in the above short exact sequence shows that X is connected. Next, for $n = 1$ we have a short exact sequence

$$0 \longrightarrow \pi_1 X \xrightarrow{f_*} \mathbb{Z} \xrightarrow{\Theta_*} \mathbb{Z} \longrightarrow 0, \tag{2.38}$$

where we have used that $\pi(\mathbf{B}\mathcal{S}) \cong \mathbb{Z}$. Since Θ_* is surjective, it is an isomorphism, and we conclude that X is simply connected.

For $n \geqslant 2$, the short exact sequences of abelian groups split by the splitting lemma, and we have isomorphisms

$$\begin{array}{ccccc} \pi_n(X \times S^1) & \xrightarrow{\cong} & \pi_n X \times \pi_n S^1 & \xrightarrow{\quad\cong\quad} & \pi_n \mathbf{B}\mathcal{S} \\ [x, y] & \longmapsto & ([x], [y]) & \longmapsto & (f_*[x])(s_*[y]). \end{array} \tag{2.39}$$

The composite isomorphism is induced by the maps $\Psi : X \times S^1 \to \mathbf{B}\mathcal{S}$ sending (x, y) to $x \cdot y$, where $\cdot$ denotes the loop product. By the above, the map Ψ is a weak equivalence, and by Whitehead's theorem we conclude that it is a homotopy equivalence. $\square$

3. Invertible TQFTs

In this chapter, we introduce *(invertible) topological quantum field theories.* The goal is to apply results from the previous chapter, to obtain our main theorem, theorem 3.6, which states (in essence) that invertible TQFT are determined by their image on the sphere. This is directly inspired by (and generalizes) a theorem of Juer and Tillmann [5, Theorem 4.3]. As a matter of technical convenience, we introduce the convenient notion of *pointed* TQFT. For the remainder of this chapter, we fix a symmetric monoidal category $\mathcal{C}$ whose unit object is denoted e.

Definition 3.1. A (2-dimensional, $\mathcal{C}$-valued) *topological quantum field theory* is a a symmetric monoidal functor from the cobordism category $\mathcal{S}$ to $\mathcal{C}$.

The $\mathcal{C}$-valued TQFTs assemble into a category $\mathbf{TQFT}_{\mathcal{C}}$, where the morphisms are symmetric monoidal natural transformations, i.e. we define

$$\mathbf{TQFT}_{\mathcal{C}} = \underline{\mathbf{SM}}(\mathcal{S}, \mathcal{C}). \tag{3.1}$$

Recall (definition B.12) that an object a of a symmetric monoidal category $\mathcal{C}$ is said to be *invertible* if there exists an object b in the category, and an isomorphism $a \otimes b \cong e$ to the unit object. The *invertible* $\mathcal{C}$-valued TQFTs can then be defined as the invertible objects of the category $\mathbf{TQFT}_{\mathcal{C}}$. For our purposes, we shall make use of an alternative definition. While showing the equivalence of these definitions is outside the scope of this text, we do provide some motivation for our preferred definition.

Let us examines what it means for a $\mathcal{C}$-valued TQFT F to be an invertible object of $\mathbf{TQFT}_{\mathcal{C}}$. By definition, F is invertible when there exists another $\mathcal{C}$-valued TQFT G and a natural isomorphism $F \otimes G \cong e_{\mathbf{TQFT}}$. This means that for every object x in $\mathcal{S}$, there is an isomorphism $Fx \otimes Gx \cong e$. In particular, hence the image Fx is an invertible object in $\mathcal{C}$. Naturality of this isomorphism means that for any morphism $f : x \to y$ in $\mathcal{S}$, the following square commutes:

$$\begin{tikzcd} Fx \otimes Gx \arrow[r, "Ff \otimes Gf"] \arrow[d, "\cong"'] & Fy \otimes Gy \arrow[d, "\cong"] \\ e \arrow[r, equal] & e, \end{tikzcd} \tag{3.2}$$

hence $Ff \otimes Gf$ is an isomorphism. By the interchange law, we have the factorization $Ff \otimes Gf = (Ff \otimes 1_{Gx}) \circ (1_{Fy} \otimes Gf)$, therefore the factor $Ff \otimes 1_{G_x}$ is an isomorphism. But the morphism $Ff \otimes 1_{Gy}$ is the image of Ff under the functor $- \otimes y$, which by lemma B.13 is an equivalence of categories. Thus the morphism Ff is an isomorphism. In conclusion, an invertible TQFT sends objects to invertible objects, and morphisms to isomorphisms, as desired.

This leads us to the following definitions.

Definition 3.2. Let $\mathcal{C}$ be a symmetric monoidal category. The *Picard groupoid* of $\mathcal{C}$, denoted $\operatorname{Pic}\mathcal{C}$, is the subcategory $\mathcal{C}$ consisting of all invertible objects and isomorphisms between them.

It is clear that $\operatorname{Pic}\mathcal{C}$ is a category. Moreover, since the subcategory $\operatorname{Pic}\mathcal{C}$ is closed under tensor product, it is in fact a symmetric monoidal category.

Definition 3.3. A $\mathcal{C}$-valued TQFT is said to be *invertible* when it takes values in the Picard groupoid $\operatorname{Pic}\mathcal{C}$.

The invertible TQFTs assemble into the category $\mathbf{SM}(\mathcal{S}, \operatorname{Pic}\mathcal{C})$, which we denote $\mathbf{TQFT}_{\mathcal{C}}^{\times}$, and is by definition

$$\mathbf{TQFT}_{\mathcal{C}}^{\times} = \mathbf{SM}(\mathcal{S}, \operatorname{Pic}\mathcal{C}). \tag{3.3}$$

Our goal is to provide a description of the category of invertible TQFTs. To this end, we introduce an intermediary notion, that of *pointed* TQFT.

3.1. Pointed TQFTs

We define pointed versions of symmetric monoidal functors and natural transformations, which behave well with respect to the unit. The extra assumptions provide some control over the data, and turn out to be insignificant up to equivalence.

Definition 3.4. A symmetric monoidal functor $(F, F_2, F_0) : \mathcal{C} \to \mathcal{C}'$ is said to be *pointed* if it preserves the unit, i.e. if $F_0 : e' \to F(e)$ is the identity morphism. A symmetric monoidal natural transformation τ is said to be *pointed* if it is the identity on the unit, i.e. if $\tau_e = 1_{e'}$.

The following proposition clarifies in what sense the pointedness assumption can be considered harmless.

Proposition 3.5. Let $\mathcal{C}'$ be a symmetric monoidal category. Then there is an equivalence of categories $\mathbf{SM}(\mathcal{C}, \mathcal{C}') \simeq \mathbf{SM}_{\bullet}(\mathcal{C}, \mathcal{C}')$.

Proof. We define a functor

$$\mathbf{SM}(\mathcal{C},\mathcal{C}') \xrightarrow{\widetilde{(-)}} \mathbf{SM}_{\bullet}(\mathcal{C},\mathcal{C}'), \tag{3.4}$$

and show that it is an equivalence of categories. Given a functor $F : \mathcal{C} \to \mathcal{C}'$ and an object a in $\mathcal{C}$, the definition is split up into cases:

$$\widetilde{F}a = \begin{cases} e', & a = e \\ Fa, & a \neq e. \end{cases} \tag{3.5}$$

For a morphism $f : a \to b$, we take advantage of the isomorphism $F_o : e' \to Fe$, and set

$$\widetilde{F}f = \begin{cases} F_0^{-1} \circ Ff \circ F_0, & a = e, b = e, \\ F_0^{-1} \circ Ff, & a \neq e, b = e, \\ Ff \circ F_0, & a = e, b \neq e, \\ Ff, & a \neq e, b \neq e. \end{cases} \tag{3.6}$$

The above ensures that $\widetilde{F}$ is pointed. It remains to define the symmetric monoidal data of a natural isomorphism $\widetilde{F}_2(a,b) : \widetilde{F}a \otimes \widetilde{F}b \to \widetilde{F}(a \otimes b)$. Since each variable can either be equal to e or not, we see an exponential increase in the number of cases. For completeness, we include the definitions below, which are simply obtained by matching the data in the only sensible way.

$a = e$	$b = e$	$a \otimes b = e$	$\widetilde{F}_2(a,b)$
0	0	0	$Fa \otimes Fb \xrightarrow{F_2} F(a \otimes b)$
0	0	1	$Fa \otimes Fb \xrightarrow{F_2} Fe \xrightarrow{F_0^{-1}} e'$
0	1	0	$Fa \otimes e' \xrightarrow{1 \otimes F_0} Fa \otimes Fe \xrightarrow{F_2} F(a \otimes e)$
1	0	0	$e' \otimes Fb \xrightarrow{F_0 \otimes 1} Fe \otimes Fb \xrightarrow{F_2} F(e \otimes b)$
1	1	0	$e' \otimes e' \xrightarrow{F_2} Fe \otimes Fe \xrightarrow{F_0} F(e \otimes e)$
0	1	1	$Fa \otimes e' \xrightarrow{1 \otimes F_0} Fa \otimes Fe \xrightarrow{F_2} F(a \otimes e) \xrightarrow{F_0^{-1}} e'$
0	0	1	$e' \otimes Fb \xrightarrow{F_0 \otimes 1} Fe \otimes Fb \xrightarrow{F_2} F(a \otimes e) \xrightarrow{F_0^{-1}} e'$
1	1	1	$e' \otimes e' \xrightarrow{F_0 \otimes F_0} Fe \otimes Fe \xrightarrow{F_2} Fe \xrightarrow{F_0^{-1}} e'$

Here we have defined $\widetilde{F}_2(a,b)$ depending on whether a, b, or $a \otimes b$ equal e. Each row in the above table represents a case, with the last column specifying the definition for $\widetilde{F_2}$ as a composite map. This completes the definition of $(\widetilde{F}, \widetilde{F}_2)$, but it still remains to show that this defines a symmetric monoidal functor. We omit this routine but tedious verification, due to the large number of cases.

Next, we define the functor $\widetilde{(-)}$ on morphisms. Given a symmetric monoidal natural transformation $\tau : F \to G$, we define its image $\widetilde{\tau}$ as follows:

$$\widetilde{\tau}_a = \begin{cases} 1_{e'}, & a = e \\ \tau_a, & a \neq e. \end{cases} \tag{3.7}$$

It is straightforward to verify that $\widetilde{\tau}$ is natural and symmetric monoidal. To see that the functor $\widetilde{(-)}$is essentially surjective, consider the natural isomorphism $\eta : \widetilde{F} \cong F$ defined by

$$\widetilde{\eta}_a = \begin{cases} F_0, & a = e \\ 1_a, & a \neq e. \end{cases} \tag{3.8}$$

This is readily verified to be a symmetric monoidal isomorphism. Finally, for the functor $\widetilde{(-)}$ to be fully faithful, it must induce a bijection of sets

$$\left\{ \begin{array}{c} \text{symmetric monoidal} \\ \text{natural} \\ \text{transformations} \\ F \to G \end{array} \right\} \cong \left\{ \begin{array}{c} \text{pointed symmetric} \\ \text{monoidal} \\ \text{natural transformations} \\ \widetilde{F} \to \widetilde{G} \end{array} \right\}, \tag{3.9}$$

for any pair of functors $F, G \in \mathbf{SM}(\mathcal{C}, \mathcal{C}')$.

To see this, note that τ can be retrieved from the data of $\widetilde{\tau}$ and the data of F and G, since $\tau_a = \widetilde{\tau}_a$ for $a \neq e$, and the value of τ_e is forced by the data of the functors F and G by the diagram

$$\begin{tikzcd} e' \arrow[r, "F_0"] \arrow[d, equal] & Fe \arrow[d, "\tau_e"] \\ e' \arrow[r, "G_0"] & Ge, \end{tikzcd} \tag{3.10}$$

i.e. we have $\tau_e = G_0 \circ F_0^{-1}$. □

3.2. The category of invertible TQFTs

Let $\mathcal{C}$ be a symmetric monoidal category with unit object e, and consider an invertible TQFT

$$(F, F_2, F_0) : \mathcal{S} \to \operatorname{Pic}\mathcal{C}. \tag{3.11}$$

Let S^2 denote the 2-sphere seen as a morphism $1 \to 1$ in $\mathcal{S}$. Then the composite morphism

$$e \xrightarrow{F_0} F(e) \xrightarrow{F_0^{-1}} e \tag{3.12}$$

is an automorphism of e. We say that this automorphism is the *the value F on the sphere*, or that it is obtained by *evaluation of F on the sphere*. The principal result of this text is the following.

Theorem 3.6. There is a symmetric equivalence of categories

$$\mathbf{TQFT}^{\times}_{\mathcal{C}} \simeq \mathrm{Aut}_{\mathcal{C}}(e). \tag{3.13}$$

Here $\mathrm{Aut}_{\mathcal{C}}(e)$ is seen as a discrete category, and equipped with a symmetric monoidal structure via composition of automorphisms. The equivalence is given by evaluation on the sphere.

Proof. Recall that G_0 denotes the group of automorphisms of 0 in $\mathcal{GS}$, and that $[S^2] \in \mathcal{GS}$ is the automorphism $q_{\mathcal{C}}(S^2)$. First, observe that the category $\underline{\mathbf{SM}_\bullet}(G_0, \mathrm{Pic}\,\mathcal{C})$ is discrete. Indeed, G_0 has only one object, so a natural transformation $\tau \in \mathrm{mor}\,\underline{\mathbf{SM}_\bullet}(G_0, \mathrm{Pic}\,\mathcal{C})$ is determined by τ_0, which is necessarily the identity.

Any symmetric monoidal functor $F : G_0 \to \mathrm{Pic}\,\mathcal{C}$ determines an automorphism of e, given by on the class of the sphere, $[S^2]$. Conversely, any automorphism of $f : e \to e$ determines a functor $F : G_0 \to \mathrm{Pic}\,\mathcal{C}$, by letting $F([S^2]) = f$. Indeed, this is the case because the group G_0 is generated by $[S^2]$, by theorem 2.15 In conclusion, we have an isomorphism of groups

$$\underline{\mathbf{SM}_\bullet}(G_0, \mathrm{Pic}\,\mathcal{C}) \cong \mathrm{Aut}_{\mathcal{C}}(e). \tag{3.14}$$

The equivalence from eq. (3.13) is now given as follows:

$$\begin{aligned}
\mathbf{TQFT}^{\times}_{\mathcal{C}} &= \mathbf{SM}(\mathcal{S}, \mathrm{Pic}\,\mathcal{C}) && \text{by definition} \\
&\cong \underline{\mathbf{SM}}(\mathcal{GS}, \mathrm{Pic}\,\mathcal{C}) && \text{by proposition 2.11} \\
&\cong \underline{\mathbf{SM}}(G_0, \mathrm{Pic}\,\mathcal{C}) && \text{by lemma 2.13} \\
&\simeq \underline{\mathbf{SM}_\bullet}(\mathbb{Z}, \mathrm{Pic}\,\mathcal{C}) && \text{by proposition 3.5} \\
&\cong \mathrm{Aut}_{\mathcal{C}}(e) && \text{by eq. (3.14)}.
\end{aligned} \tag{3.15}$$

By the discussion leading to eq. (3.14), we see that the equivalence is given by evaluation on the sphere. By examining the symmetric monoidal structure on $\mathbf{TQFT}^{\times}_{\mathcal{C}}$ remark B.11, it is straightforward to show that the composite of the above functors is in fact symmetric monoidal, so that we have a symmetric monoidal equivalence as desired. □

A. Strict 2-categories

We define *2-categories* as Cat-enriched categories, i.e. we consider only *strict* 2-categories. The framework of 2-categories provides a general notion of equivalence between objects. We show that equivalent objects give rise to equivalent hom-categories.

A.1. 2-Categories

Informally, a 2-category is a category equipped with a notion of “morphisms between morphisms”. All available morphisms can be composed in an associative and unital way, which is what the following definition expresses.

Definition A.1. A *2-category* $\underline{\mathbf{C}}$ consists of:

- a collection $\operatorname{obj}\underline{\mathbf{C}}$ of *objects* of $\underline{\mathbf{C}}$,
- for each pair of objects $x, y \in \operatorname{obj}\underline{\mathbf{C}}$, a small category $\underline{\mathbf{C}}(x,y)$,
- for each object $x \in \operatorname{obj}\underline{\mathbf{C}}$, a functor $1_x : * \to \underline{\mathbf{C}}(x,x)$,
- for each triple $x, y, z \in \operatorname{obj}\underline{\mathbf{C}}$, a functor

$$\underline{\mathbf{C}}(y,z) \times \underline{\mathbf{C}}(x,y) \xrightarrow{\circ} \underline{\mathbf{C}}(x,z), \tag{A.1}$$

such that the following diagrams commute for all $x, y, z, w \in \operatorname{obj}\underline{\mathbf{C}}$:

i) *(Associativity)*

$$\begin{tikzcd}
\underline{\mathbf{C}}(z,w) \times \underline{\mathbf{C}}(y,z) \times \underline{\mathbf{C}}(x,y) \arrow[r, "1\times\circ"] \arrow[d, "\circ\times 1"'] & \underline{\mathbf{C}}(z,w) \times \underline{\mathbf{C}}(x,z) \arrow[d, "\circ"] \\
\underline{\mathbf{C}}(y,w) \times \underline{\mathbf{C}}(x,y) \arrow[r, "\circ"'] & \underline{\mathbf{C}}(x,w),
\end{tikzcd} \tag{A.2}$$

ii) *(Unitality)*

$$\begin{tikzcd}
\underline{\mathbf{C}}(x,y) \times * \arrow[r, "1\times 1_x"] \arrow[dr, "1"'] & \underline{\mathbf{C}}(x,y) \times \underline{\mathbf{C}}(x,x) \arrow[d, "\circ"] \\
& \underline{\mathbf{C}}(x,y),
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbf{C}}(y,y) \times \underline{\mathbf{C}}(x,y) \arrow[d, "\circ"'] & * \times \underline{\mathbf{C}}(x,y) \arrow[l, "1_y\times 1"'] \arrow[dl, "1"] \\
\underline{\mathbf{C}}(x,y). &
\end{tikzcd} \tag{A.3}$$

The objects of $\underline{\mathbf{C}}$ will also be called *0-cells*. The objects of the *hom-categories* $\underline{\mathbf{C}}(x, y)$ are called *1-cells* and are thought of as morphisms between 0-cells. Accordingly, we shall denote an object $F \in \underline{\mathbf{C}}(x, y)$ by

$$F : x \longrightarrow y \tag{A.4}$$

The functor $\circ$ provides a means of composing 1-cells, and the functor $1_x : * \to \underline{\mathbf{C}}(x, x)$ singles out a 1-cell (also denoted 1_x) which acts as the identity under this composition. With this data, the set of 0-cells is a category, where the morphisms are 1-cells.

Morphisms $\tau : F \to G$ between 1-cells are called *2-cells* and are depicted as follows:

$$\begin{tikzcd} x \arrow[r, "F"] \arrow[d, no head] & y \arrow[d, no head] \\ x \arrow[r, "G"'] \arrow[ur, phantom, "\Downarrow\tau"] & y. \end{tikzcd} \tag{A.5}$$

We see that there are two ways to compose 2-cells. First, using ordinary composition in the category $\underline{\mathbf{C}}(x, y)$, a pair of 2-cells $\tau : F \to G$ and $\sigma : G \to H$ can be composed to a 2-cell $\sigma \cdot \tau : F \to H$. This composite will be called their *vertical composition*, and depicted

$$\begin{tikzcd} x \arrow[r, "F"] \arrow[d, no head] & y \arrow[d, no head] \\ x \arrow[r, "G"'] \arrow[ur, phantom, "\Downarrow\tau"] \arrow[d, no head] & y \arrow[d, no head] \\ x \arrow[r, "H"'] \arrow[ur, phantom, "\Downarrow\sigma"] & y. \end{tikzcd} \tag{A.6}$$

Second, by functoriality of $\circ$, a pair of 2-cells $\tau \in \operatorname{mor} \underline{\mathbf{C}}(x, y)$ and $\tau' \in \operatorname{mor} \underline{\mathbf{C}}(y, z)$ can be "composed" to a 2-cell $\tau' \circ \tau \in \operatorname{mor} \underline{\mathbf{C}}(x, z)$. This 2-cell will be called their *horizontal composition*, and depicted

$$\begin{tikzcd} x \arrow[r, "F"] \arrow[d, no head] & y \arrow[r, "F'"] \arrow[d, no head] & z \arrow[d, no head] \\ x \arrow[r, "G"'] \arrow[ur, phantom, "\Downarrow\tau"] & y \arrow[r, "G'"'] \arrow[ur, phantom, "\Downarrow\tau'"] & z. \end{tikzcd} \tag{A.7}$$

The object 1_x of $\underline{\mathbf{C}}(x, x)$ admits an identity 2-cell denoted 1_{1_x}, which is an identity with respect to vertical composition. In addition, the diagrams in definition A.1 imply that 1_{1_x} is an identity with respect to horizontal composition. With these notions of identities and composition, we have the obvious notions of inverse and isomorphism.

Remark A.2. It is natural to wonder how the two ways of composing 2-cells interact. A priori, the following picture can be interpreted in two ways,

τ	τ'
σ	σ'

depending whether one uses horizontal composition first and vertical composition second, or vice versa. Functoriality of $\circ$ ensures that this is unambiguous, i.e.

$$(\sigma' \cdot \tau') \circ (\sigma \cdot \tau) = (\sigma' \circ \sigma) \cdot (\tau' \circ \tau).$$

This is known the *interchange law.*

Example A.3. The prototypical 2-category is the category of small categories, where the 0-cells are categories, 1-cells are functors, and 2-cells are natural transformations. When thought of as a 2-category, the category of small categories is denoted $\underline{\mathbf{Cat}}$.

First, for each pair of categories $\mathcal{X}$ and $\mathcal{Y}$, there is a category $\underline{\mathbf{Cat}}(\mathcal{X}, \mathcal{Y})$ whose objects are functors $\mathcal{X} \to \mathcal{Y}$ and morphisms are natural transformations. The identity natural transformation is pointwise made up of identities, and the (vertical) composition of natural transformations is given as follows. Given functors and natural transformations

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] & \mathcal{Y}, \end{tikzcd} \qquad \Downarrow\tau,\ \Downarrow\sigma \tag{A.8}$$

the vertical composite $\sigma \cdot \tau$ is defined for an object x in $\mathcal{X}$ by $(\sigma \cdot \tau)_x = \sigma_x \tau_x$, which is natural because σ and τ are. The composition in $\underline{\mathbf{Cat}}(\mathcal{X}, \mathcal{Y})$ is associative and unital since it is contructed from the composition in the category $\mathcal{Y}$.

Second, for any functor $F : \mathcal{X} \to \mathcal{X}$, there is an obvious functor $1_F : * \to \underline{\mathbf{Cat}}(\mathcal{X}, \mathcal{Y})$ picking F and the identity natural transformation of F.

Third, we have an assignment

$$\underline{\mathbf{Cat}}(\mathcal{Y}, \mathcal{Z}) \times \underline{\mathbf{Cat}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\ \circ\ } \underline{\mathbf{Cat}}(\mathcal{X}, \mathcal{Z}) \tag{A.9}$$

sending functors to their ordinary composite, and natural transformations

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F"{name=A}] \arrow[r, bend right, "G"'{name=B}] & \mathcal{Y} \arrow[r, bend left, "F'"{name=C}] \arrow[r, bend right, "G'"'{name=D}] & \mathcal{Z}, \arrow[from=A, to=B, Rightarrow, "\tau"] \arrow[from=C, to=D, Rightarrow, "\tau'"] \end{tikzcd} \tag{A.10}$$

to their horizontal composite $\tau' \circ \tau$ defined as follows. For any object x in $\mathcal{X}$, we let $(\tau' \circ \tau)_x : F'Fx \to G'Gx$ be either composite in the diagram below:

$$\begin{tikzcd} F'Fx \arrow[r, "F'(\tau_x)"] \arrow[d, "\tau'_{Fx}"'] & F'Gx \arrow[d, "\tau'_{Gx}"] \\ G'Fx \arrow[r, "G'(\tau_x)"'] & G'Gx. \end{tikzcd} \tag{A.11}$$

(The diagram commutes by naturality of τ'.) Let us check that $\tau' \circ \tau$ is natural. For any morphism $f : a \to b$ in $\mathcal{X}$, the perimeter of the following diagram commutes,

$$\begin{tikzcd} F'Fa \arrow[r, "F'Ff"] \arrow[d, "F'(\tau_a)"] \arrow[dd, bend right, "(\tau' \circ \tau)_a"'] & F'Fb \arrow[d, "F'(\tau_b)"'] \arrow[dd, bend left, "(\tau' \circ \tau)_b"] \\ F'Ga \arrow[r, "F'Gf"] \arrow[d, "\tau'_{Ga}"] & F'Fb \arrow[d, "\tau'_{Gb}"'] \\ G'Ga \arrow[r, "G'Gf"'] & G'Gb, \end{tikzcd} \tag{A.12}$$

since the upper square is obtained by applying the functor F' to the naturality square for τ relative to f, and the lower square is obtained by the naturality of τ' relative to Gf. Thus the assignment $\circ$ makes sense, and functoriality of $\circ$ follows from the functoriality of F, F', G, and G'.

Finally, the functor $\circ$ is associative and unital in the sense of definition A.1 because it was defined in terms of unital and associative operations. This concludes the construction of $\underline{\mathbf{Cat}}$.

A.2. Equivalences in 2-categories

The existence of 2-cells allows us to establish a notion weaker than isomorphism.

Definition A.4. Given a pair of objects x, y in a 2-category $\underline{\mathbf{C}}$, an *equivalence* between x and y is a pair of 1-morphisms

$$\begin{tikzcd} x \arrow[r, bend left, "S"] & y \arrow[l, bend left, "T"] \end{tikzcd} \tag{A.13}$$

together with a pair of 2-isomorphisms $\eta : \mathrm{id}_x \cong T \circ S$ and $\varepsilon : S \circ T \cong \mathrm{id}_y$.

Our goal is to show that an equivalence of objects induces an equivalence of hom-categories. Along the way, we show that 1-cells induce functors (lemma A.5) and 2-cells induce natural transformations (lemma A.6).

Lemma A.5. Let $F : x \to y$ be a 1-cell in a 2-category $\underline{\mathbf{C}}$. Then there exists a functor $\overline{F} : \underline{\mathbf{C}}(y,z) \to \underline{\mathbf{C}}(x,z)$.

Proof. The functor $\overline{F}$ is defined as the composite functor

$$\underline{\mathbf{C}}(y,z) \xrightarrow{\cong} \underline{\mathbf{C}}(y,z) \times * \xrightarrow{1\times F} \underline{\mathbf{C}}(y,z) \times \underline{\mathbf{C}}(x,y) \xrightarrow{\circ} \underline{\mathbf{C}}(x,z), \tag{A.14}$$

where F is seen as a functor $* \to \underline{\mathbf{C}}(x,y)$. Concretely, the functor $\overline{F}$ acts by precomposition. Given a functor $F' \in \operatorname{obj} \mathcal{C}(y,z)$ and a natural transformation $\tau' \in \operatorname{mor} \mathcal{C}(y,z)$, we have

$$\overline{F}(F') = F' \circ F \tag{A.15}$$
$$\overline{F}(\tau') = \tau' \circ 1_F. \tag{A.16}$$

Pictorially, we have

$$\begin{array}{|c|}\hline \overline{F}(\tau') \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1_F & \tau' \\ \hline \end{array}\,. \tag{A.17}$$

□

Lemma A.6. Let $\tau : F \to G$ be a 2-cell between 1-cells $F, G : x \to y$ in a 2-category $\underline{\mathbf{C}}$. Then there exists a natural transformation $\overline{\tau} : \overline{F} \to \overline{G}$ between the pair of functors $\overline{F}, \overline{G} : \underline{\mathbf{C}}(y,z) \to \underline{\mathbf{C}}(x,z)$.

Proof. The component of $\overline{\tau}$ corresponding to an object F' of $\underline{\mathbf{C}}(y,z)$ is defined by

$$\overline{\tau}_{F'} = 1_{F'} \circ \tau. \tag{A.18}$$

$$\begin{array}{|c|}\hline \overline{\tau}_{F'} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \tau & 1_{F'} \\ \hline \end{array}\,. \tag{A.19}$$

The naturality square of $\overline{\tau}$ takes the following form. For any pair of 1-cells $F', G' \in \operatorname{obj} \underline{\mathbf{C}}(y,z)$, and for any 2-cell $\tau' : F' \to G'$, the following diagram should commute:

$$\begin{tikzcd}
\overline{F}(F') \arrow[r, "\overline{F}(\tau')"] \arrow[d, "\overline{\tau}_{F'}"'] & \overline{F}(G') \arrow[d, "\overline{\tau}_{G'}"] \\
\overline{G}(F') \arrow[r, "\overline{G}(\tau')"'] & \overline{G}(G').
\end{tikzcd} \tag{A.20}$$

For the composite (↳), we have (by definition and the interchange law):

$$\overline{G}(\tau') \cdot \overline{\tau}_{F'} = (\tau' \circ 1_G) \cdot (1_{F'} \circ \tau) = (\tau' \cdot 1_F) \circ (1_G \cdot \tau) = \tau' \circ \tau, \tag{A.21}$$

and for the composite (↴), we similarly have

$$\overline{\tau}_{G'} \cdot \overline{F}(\tau') = (1_{G'} \circ \tau) \cdot (\tau' \circ 1_F) = (1_F \cdot \tau) \circ (\tau' \cdot 1_G) = \tau' \circ \tau, \tag{A.22}$$

and so the diagram commutes as desired. Visually, the above computation is as follows:

$$\begin{array}{|c|}\hline \overline{G}(\tau') \\ \hline \overline{\tau}_{F'} \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1_G & \tau' \\ \hline \tau & 1_{F'} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \tau & \tau' \\ \hline \end{array} = \begin{array}{|c|c|}\hline \tau & 1_{F'} \\ \hline 1_G & \tau' \\ \hline \end{array} = \begin{array}{|c|}\hline \overline{\tau}_{G'} \\ \hline \overline{F}(\tau') \\ \hline \end{array} . \tag{A.23}$$

□

It is easy to see that the mapping $F \mapsto \overline{F}, \tau \mapsto \overline{\tau}$ respects identities and composition. In particular it takes isomorphisms to isomorphisms. We deduce the following:

Proposition A.7. Let x and y be equivalent objects in a 2-category $\underline{\mathbf{C}}$. Then there exists an equivalence of categories between $\underline{\mathbf{C}}(y, z)$ and $\underline{\mathbf{C}}(x, z)$, for any object z.

The above proposition shows that the first variable in a hom-category $\underline{\mathbf{C}}(x, y)$ can be swapped for an equivalent one. By dual arguments, the same holds true for the second variable.a 2 This can be seen as a 2-categorical analog of the fact that isomorphic objects induce an isomorphism of representable functors, which follows from the Yoneda lemma. While there exists a 2-categorical version of the Yoneda lemma, we have chosen to present proposition A.7 in an elementary way.

B. Symmetric monoidal gadgetry

We introduce symmetric monoidal categories, functors and natural transformations, and verify that these make up the data of a 2-category $\underline{\mathrm{SM}}$.

B.1. Categories, functors, and natural transformations

Symmetric monoidal categories

Symmetric monoidal categories capture the idea of "multiplying" objects and morphisms, as exemplified by the tensor product of vector spaces:

$$(A, B) \mapsto A \otimes B.$$

Depending on the choice of parenthesization, there are two ways to form the tensor products of three vector spaces. The two resulting vector spaces are not equal in general, but only naturally isomorphic:

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C.$$

Increasing the number of arguments further yields many more parenthesizations, but also more isomorphisms between the resulting objects. Luckily, these isomorphisms are compatible: any conceivable diagram made up of them commutes. This phenomenon is known as *coherence*, and is not unique to the category of vector spaces. It turns out that coherence be deduced from the commutativity of a few simple diagrams, on which the definition of a symmetric monoidal category is modelled.

Definition B.1. A *symmetric monoidal category* $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ consists of:

- A category $\mathcal{C}$.
- An object $e \in \operatorname{obj} \mathcal{C}$, called the *unit object*.
- A functor called the *tensor product*

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}, \tag{B.1}$$

  i.e. for each pair of objects a, b in $\mathcal{C}$, an object $\otimes(b, a) = a \otimes b$, and for each pair of morphisms $f : a \to b$ and $f' : a' \to b'$ in $\mathcal{C}$, a morphism $f \otimes f' : a \otimes a' \to b \otimes b'$.

- A natural isomorphism α called the *associator*:

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}\cong"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C},
\end{tikzcd}
\tag{B.2}
$$

  i.e. isomorphisms $\alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$ natural in $a, b, c \in \operatorname{obj} \mathcal{C}$.

- Natural isomorphisms λ and ρ called the *left* and *right unitor*, respectively:

$$
\begin{tikzcd}
\mathcal{C} \times * \arrow[r, "1 \times e"] \arrow[dr, bend right, "\mathrm{pr}_1"'] \arrow[r, phantom, shift right=6ex, "\overset{\lambda}{\Swarrow}\cong"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \arrow[r, phantom, shift right=6ex, "\cong\overset{\rho}{\Searrow}"] & * \times \mathcal{C} \arrow[l, "e \times 1"'] \arrow[dl, bend left, "\mathrm{pr}_2"] \\
 & \mathcal{C}. &
\end{tikzcd}
\tag{B.3}
$$

  i.e. isomorphisms $\lambda_a : e \otimes a \cong a$ and $\rho_a : a \otimes e \cong a$, natural in $a \in \operatorname{obj} \mathcal{C}$.

- A natural isomorphism γ called the *braiding*:

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[rr, "\otimes"] \arrow[dr, "\mathrm{twist}"'] \arrow[rr, phantom, shift right=5ex, "\gamma \Downarrow \cong"] & & \mathcal{C} \\
 & \mathcal{C} \times \mathcal{C}, \arrow[ur, "\otimes"'] &
\end{tikzcd}
\tag{B.4}
$$

  i.e. isomorphisms $\gamma_{a,b} : a \otimes b \cong b \otimes a$, natural in $a, b \in \operatorname{obj} \mathcal{C}$.

The above data are required to satisfy the following properties.

i) *(Pentagon axiom)* For all objects a, b, c of $\mathcal{C}$, the following diagram commutes:

$$
\begin{tikzcd}
 & (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha"] & \\
a \otimes (b \otimes (c \otimes d)) \arrow[ur, "\alpha"] \arrow[d, "1 \otimes \alpha"'] & & ((a \otimes b) \otimes c) \otimes d \\
a \otimes ((b \otimes c) \otimes d) \arrow[rr, "\alpha"'] & & (a \otimes (b \otimes c)) \otimes d. \arrow[u, "\alpha \otimes 1"']
\end{tikzcd}
\tag{B.5}
$$

ii) *(Triangle axiom)* For every object a of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
a \otimes (e \otimes c) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes\lambda"'] & & (a \otimes e) \otimes c \arrow[dl, "\rho\otimes 1"] \\
& a \otimes c. &
\end{tikzcd} \tag{B.6}$$

Moreover, $\lambda_{e,e} = \rho_{e,e} : e \otimes e \to e$.

iii) *(Symmetry axiom)* For all objects a, b of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
a \otimes b \arrow[rr, "1"] \arrow[dr, "\gamma_{a,b}"'] & & a \otimes b \\
& b \otimes a. \arrow[ur, "\gamma_{b,a}"'] &
\end{tikzcd} \tag{B.7}$$

iv) *(Triangle braiding axiom)* For any object a of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
a \otimes e \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & e \otimes a \arrow[dl, "\lambda"] \\
& a. &
\end{tikzcd} \tag{B.8}$$

v) *(Hexagon braiding axiom)* For all objects a, b, c of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\gamma"] \arrow[d, "\alpha^{-1}"'] & c \otimes (a \otimes b) \arrow[d, "\alpha"] \\
a \otimes (b \otimes c) \arrow[d, "1\otimes\gamma"'] & (c \otimes a) \otimes b \arrow[d, "\gamma\otimes 1"] \\
a \otimes (c \otimes b) \arrow[r, "\alpha"'] & (a \otimes c) \otimes b.
\end{tikzcd} \tag{B.9}$$

We adopt a few conventions to ease notation. The symbol $\otimes$ will be overloaded to simultaneously denote the tensor product of objects and morphisms in different categories. For conciseness, we say that $\mathcal{C}$ is a symmetric monoidal category, leaving the extra data implicit.

Example B.2. Let M be an abelian monoid, seen as a category with only one object $*$. Then M can be given the structure of a symmetric monoidal category as follows. The tensor product of morphisms f and g in M is given by their composition, i.e. we let $f \otimes g = fg$. For the unique object $*$, we let $* \otimes * = *$.

The associator, unitors and braiding are all identities.

When the associator, unitors and braiding are identities, we say that the symmetric monoidal category is *strict.*

Proposition B.3. In conclusion, the monoid $\mathrm{End}_{\mathcal{C}}(e)$ is abelian.

Proof. Let $\mathcal{C}$ be a symmetric monoidal category, and let f, g, h, and i be morphisms in $\mathcal{C}$. The functoriality of the tensor product $\otimes$ implies that

$$(f \otimes g) \circ (h \otimes i) = (f \circ h) \otimes (g \circ i), \tag{B.10}$$

whenever the compositions make sense. This equation known as the *interchange law.* Let $f : e \to e$ be an endomorphism of the unit object of $\mathcal{C}$. Naturality of ρ implies that the following diagram commutes:

$$\begin{tikzcd} e \otimes e \arrow[r, "f\otimes 1"] \arrow[d, "\rho"'] & e \otimes e \arrow[d, "\rho"] \\ e \arrow[r, "f"] & e \\ e \otimes e \arrow[r, "1\otimes f"] \arrow[u, "\lambda"] & e \otimes e, \arrow[u, "\lambda"'] \end{tikzcd} \tag{B.11}$$

but by the triangle axiom, $\lambda_e = \rho_e$, hence we have $f \otimes 1 = 1 \otimes f$. Let $g : e \to e$ be another endomorphism of the unit object, and write 1 for 1_e. Then

$$\begin{aligned} f \otimes g &= (f \circ 1) \otimes (1 \circ g) \\ &= (f \otimes 1) \circ (1 \otimes g) \quad \text{by the interchange law} \\ &= (1 \otimes f) \circ (g \otimes 1) \\ &= (1 \circ g) \otimes (f \circ 1) \quad \text{by the interchange law} \\ &= g \otimes f. \end{aligned} \tag{B.12}$$

Moreover, by the naturality of λ, and the equality $\lambda_e = \rho_e$, we have the following commutative diagram:

$$\begin{tikzcd} e \otimes e \arrow[r, "1\otimes f"] \arrow[rr, bend left, "g\otimes f"] & e \otimes e \arrow[r, "g\otimes 1"] \arrow[d, "\lambda"] & e \otimes e \arrow[d, "\lambda"] \\ e \arrow[u, "\lambda^{-1}"] \arrow[r, "f"'] & e \arrow[r, "g"'] & e, \end{tikzcd} \tag{B.13}$$

and a similar diagram where the roles of f and g are exchanged. Thus we have

$$\begin{aligned} g \circ f &= \lambda \circ (g \otimes f) \circ \lambda^{-1} \\ &= \lambda \circ (f \otimes g) \circ \lambda^{-1} \\ &= f \circ g. \end{aligned} \tag{B.14}$$

$\square$

Symmetric monoidal functors

Definition B.4. Given monoidal categories $\mathcal{C}$ and $\mathcal{C}'$, a *symmetric monoidal functor* $(F, F_2, F_0) : \mathcal{C} \to \mathcal{C}'$ consists of:

- A functor $F : \mathcal{C} \to \mathcal{C}'$.
- A natural isomorphism:
$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "F \times F"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "F_2 \swarrow \cong"] & \mathcal{C}' \times \mathcal{C}' \arrow[d, "\otimes"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{C}', \end{tikzcd} \tag{B.15}$$
i.e. isomorphisms $F_2(a,b) : Fa \otimes Fb \to F(a \otimes b)$, natural in $a, b \in \operatorname{obj} \mathcal{C}$.
- An isomorphism:
$$F_0 : e' \xrightarrow{\cong} Fe. \tag{B.16}$$

The above data are required to satisfy the following properties.

i) *(Hexagon axiom)* For all objects a, b, and c in $\mathcal{C}$, the following diagram commutes:
$$\begin{tikzcd} Fa \otimes (Fb \otimes Fc) \arrow[r, "\alpha'"] \arrow[d, "1 \otimes F_2"'] & (Fa \otimes Fb) \otimes Fc \arrow[d, "F_2 \otimes 1"] \\ Fa \otimes F(b \otimes c) \arrow[d, "F_2"'] & (Fa \otimes b) \otimes Fc \arrow[d, "F_2"] \\ F((a \otimes (b \otimes c)) \arrow[r, "F\alpha"'] & F((a \otimes b) \otimes c). \end{tikzcd} \tag{B.17}$$

ii) *(Left- and right-square axioms)* For every object a in $\mathcal{C}$, the following diagrams commute:
$$\begin{tikzcd} e' \otimes Fa \arrow[r, "\lambda'"] \arrow[d, "F_0 \otimes 1"'] & Fa \\ Fe \otimes Fa \arrow[r, "F_2"'] & F(e \otimes a), \arrow[u, "F\lambda"'] \end{tikzcd} \qquad \begin{tikzcd} Fa \otimes e' \arrow[r, "\rho'"] \arrow[d, "1 \otimes F_0"'] & Fa \\ Fa \otimes Fe \arrow[r, "F_2"'] & F(a \otimes e). \arrow[u, "F\rho"'] \end{tikzcd} \tag{B.18}$$

iii) *(Symmetry axiom)* For every object a in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} Fa \otimes Fb \arrow[r, "\gamma'"] \arrow[d, "F_2"'] & Fb \otimes Fa \arrow[d, "F_2"] \\ F(a \otimes b) \arrow[r, "F\gamma"'] & F(b \otimes a). \end{tikzcd} \tag{B.19}$$

For conciseness, we sometimes refer to the entire triple (F, F_2, F_0) simply by F.

Lemma B.5. The composition of symmetric monoidal functors is a symmetric monoidal functor.

Proof. Given a pair of composable symmetric monoidal functors

$$\mathcal{C} \xrightarrow{\ F\ } \mathcal{C}' \xrightarrow{\ G\ } \mathcal{C}'', \tag{B.20}$$

we can make the composite functor $GF : \mathcal{C} \to \mathcal{C}''$ into a symmetric monoidal functor by defining $(GF)_0$ and $(GF)_2$ as follows:

$$\begin{tikzcd} e'' \arrow[rr, "(GF)_0"] \arrow[dr, "G_0"'] & & GFe \\ & Ge', \arrow[ur, "G(F_0)"'] & \end{tikzcd} \qquad \begin{tikzcd} GFa \otimes GFb \arrow[rr, "(GF)_2"] \arrow[dr, "G_2"'] & & GF(a \otimes b) \\ & G(Fa \otimes Fb). \arrow[ur, "G(F_2)"'] & \end{tikzcd} \tag{B.21}$$

Let us verify that the resulting $(GF, (GF)_2, (GF)_0)$ is a symmetric monoidal functor. Naturality (in the second variable) of $(GF)_2$ relative to a morphism $f : b \to c$ in $\mathcal{C}$ amounts to the commutativity of the perimeter of the diagram below:

$$\begin{tikzcd} GFa \otimes GFb \arrow[r, "G_2"] \arrow[d, "1 \otimes GFf"'] \arrow[rr, bend left, "(GF)_2"] & G(Fa \otimes Fb) \arrow[r, "G(F_2)"] \arrow[d, "G(1 \otimes Ff)"'] & GF(a \otimes b) \arrow[d, "GF(1 \otimes f)"] \\ GFa \otimes GFc \arrow[r, "G_2"'] \arrow[rr, bend right, "(GF)_2"'] & G(Fa \otimes Fc) \arrow[r, "G(F_2)"'] & GF(a \otimes c). \end{tikzcd} \tag{B.22}$$

Here, the left-hand square by naturality of G_2 relative to the morphism Ff, and the right-hand square commutes by applying G to the naturality square for F. The rest of the diagram commutes by definition of $(GF)_2$. Naturality in the first variable is checked similarly.

The hexagon axiom for GF amounts to the commutativity of the perimeter of the diagram below:

$$\begin{tikzcd}
GFa \otimes (GFb \otimes GFc) \arrow[rrrrr, "\alpha''"] \arrow[dd, "1\otimes(GF)_2"'] \arrow[dr, "1\otimes G_2"] & & & & & (GFa \otimes GFb) \otimes GFc \arrow[dl, "G_2\otimes 1"'] \arrow[dd, "(GF)_2\otimes 1"] \\
& GFa \otimes G(Fb \otimes Fc) \arrow[dl, "1\otimes G(F_2)"] \arrow[dr, "G_2"] & & & G(Fa \otimes Fb) \otimes GFc \arrow[dl, "G_2"'] \arrow[dr, "G(F_2)\otimes 1"'] & \\
GFa \otimes GF(b \otimes c) \arrow[dd, "(GF)_2"'] \arrow[dr, "G_2"] & & G(Fa \otimes (Fb \otimes Fc)) \arrow[r, "G\alpha'"] \arrow[dl, "G(1\otimes F_2)"] & G((Fa \otimes Fb) \otimes Fc) \arrow[dr, "G(F_2\otimes 1)"'] & & GF(a \otimes b) \otimes GFc \arrow[dl, "G_2"'] \arrow[dd, "(GF)_2"] \\
& G(Fa \otimes F(b \otimes c)) \arrow[dl, "G(F_2)"] & & & G(F(a \otimes b) \otimes Fc) \arrow[dr, "G(F_2)"'] & \\
GF(a \otimes (b \otimes c)) \arrow[rrrrr, "GF\alpha"'] & & & & & GF((a \otimes b) \otimes c).
\end{tikzcd} \tag{B.23}$$

Here the lower hexagon is obtained by applying the functor G to the hexagonal diagram for F. The upper hexagon is obtained by taking the hexagon diagram for G with objects Fa, Fb, and Fc. The triangular diagrams commute by definition of $(GF)_2$, and the square diagrams commute by naturality of G_2.

The left-square axiom for GF amounts to the commutativity of the perimeter of the diagram below:

$$
\begin{tikzcd}
GFb \otimes e'' \arrow[rrr, "\rho''"] \arrow[dr, "1\otimes G_0"] \arrow[ddd, "1\otimes (GF)_0"'] & & & GFb \\
& GFb \otimes Ge' \arrow[r, "G_2"] \arrow[d, "1\otimes G(F_0)"'] & G(Fb \otimes e') \arrow[ur, "G\rho'"] \arrow[d, "G(1\otimes F_0)"] & \\
& GFb \otimes GFe \arrow[r, "G_2"'] \arrow[dl, equal] & G(Fb \otimes Fe) \arrow[dr, "G(F_2)"] & \\
GFb \otimes GFe \arrow[rrr, "(GF)_2"'] & & & GF(b \otimes e). \arrow[uuu, "GF(\rho)"']
\end{tikzcd}
\tag{B.24}
$$

Here the right trapezoid is obtained by applying G to the left-square diagram for F. The upper trapezoid diagram is obtained by taking the square diagram for G with the object Fa. The middle square commutes by naturality of G_2. The remaining part of the diagram commutes by definition of $(GF)_2$. The right-square axiom is verified similarly.

The symmetry axiom for GF amounts to the commutativity of the perimeter of the diagram below:

$$
\begin{tikzcd}
(GFa) \otimes (GFb) \arrow[r, "\gamma''"] \arrow[d, "G_2"'] & (GFb) \otimes (GFa) \arrow[d, "G_2"] \\
G(Fa \otimes Fb) \arrow[r, "G(\gamma')"'] \arrow[d, "G(F_2)"'] & G(Fb \otimes Fa) \arrow[d, "G(F_2)"] \\
GF(a \otimes b) \arrow[r, "GF(\gamma)"'] & GF(b \otimes a).
\end{tikzcd}
\tag{B.25}
$$

Here the lower square is obtained by applying the functor G to the symmetry square for F, and the upper square commutes by the symmetry square of G (in the objects Fa and Fb). $\square$

The composition of symmetric monoidal functors just defined is clearly associative. Thus there is a category whose objects are symmetric monoidal categories, and whose morphisms are symmetric monoidal functors. We denote this category SM.

Example B.6. Let $\mathcal{C}$ be a symmetric monoidal category and let M be the full subcategory of $\mathcal{C}$ with a single object, the unit object e. By appendix B.1, M is an abelian monoid, hence composition defines a symmetric monoidal structure on M, as in example B.2.

Let us equip the inclusion $F : M \hookrightarrow \mathcal{C}$ with the structure of a symmetric monoidal functor. We let $F_2 = \lambda = \rho : e \otimes e \to e$ and $F_0 = 1 : e \to e$. Naturality of F_2 with respect to $f, g : e \to e$ amounts to the commutativity of the perimeter of the diagram below:

$$\begin{tikzcd}
e \otimes e \arrow[r, "1\otimes f"] \arrow[rr, bend left, "g\otimes f"] \arrow[d, "\lambda"'] & e \otimes e \arrow[r, "g\otimes 1"] \arrow[d, "\lambda=\rho"] & e \otimes e \arrow[d, "\rho"] \\
e \arrow[r, "f"'] & e \arrow[r, "g"'] & e.
\end{tikzcd} \tag{B.26}$$

Here each square commutes by naturality of λ, and the top triangle commutes by the interchange law (see appendix B.1). Next we check that F is a symmetric monoidal functor. The hexagon axiom for F amounts to the commutativity of the perimeter of the diagram below:

$$\begin{tikzcd}
e \otimes (e \otimes e) \arrow[r, "\alpha"] \arrow[d, "1\otimes\lambda"'] & (e \otimes e) \otimes e \arrow[d, "\lambda\otimes 1"] \\
e \otimes e \arrow[r, "1"] \arrow[d, "\lambda"'] & e \otimes e \arrow[d, "\lambda"] \\
e \arrow[r, "1"] & e.
\end{tikzcd} \tag{B.27}$$

Here the top square commutes by the triangle axiom for $\mathcal{C}$, and the bottom square commutes trivially. The square axioms for F hold trivially, since we chose $F_2 = \lambda = \rho$, and the symmetry axiom for F holds by the triangle braiding axiom for $\mathcal{C}$.

In conclusion, there is a symmetric monoidal functor $M \hookrightarrow \mathcal{C}$.

Symmetric monoidal natural transformations

Having defined functors (1-cells), the next step is to define natural transformations (2-cells) in a symmetric monoidal context.

Definition B.7. Let F and G be symmetric monoidal functors $\mathcal{C} \to \mathcal{C}'$. A *symmetric monoidal natural transformation* $\tau : F \to G$ is a natural transformation $\tau : F \to G$ such that the following diagrams commute:

i) *(Unit axiom)*

$$\begin{tikzcd} e' \arrow[r, "F_0"] \arrow[d, equal] & Fe \arrow[d, "\tau_e"] \\ e' \arrow[r, "G_0"'] & Ge. \end{tikzcd} \tag{B.28}$$

ii) *(Tensor axiom)*

$$\begin{tikzcd} Fa \otimes Fb \arrow[r, "F_2"] \arrow[d, "\tau_a \otimes \tau_b"'] & F(a \otimes b) \arrow[d, "\tau_{a \otimes b}"] \\ Ga \otimes Gb \arrow[r, "G_2"'] & G(a \otimes b). \end{tikzcd} \tag{B.29}$$

Remark B.8. The use of the word *symmetric* in the above definition is justified without further requirements, since

$$\begin{tikzcd} Fa \otimes Fb \arrow[r, "\gamma'"] \arrow[d, "\tau_a \otimes \tau_b"'] & Fb \otimes Fa \arrow[d, "\tau_b \otimes \tau_a"] \\ Ga \otimes Gb \arrow[r, "\gamma'"'] & Gb \otimes Ga \end{tikzcd} \tag{B.30}$$

commutes by naturality of γ' (relative to the morphisms τ_a and τ_b), and

$$\begin{tikzcd} F(a \otimes b) \arrow[r, "F\gamma"] \arrow[d, "\tau_{a \otimes b}"'] & F(b \otimes a) \arrow[d, "\tau_{b \otimes a}"] \\ G(a \otimes b) \arrow[r, "G\gamma"'] & G(b \otimes a) \end{tikzcd} \tag{B.31}$$

commutes by naturality of τ (relative to the morphism γ).

Next we show that symmetric monoidal transformations can be composed. As with ordinary natural transformations, we can consider two composition laws: vertical and horizontal (see example A.3 for the way these were defined).

Lemma B.9. Let τ and σ be symmetric monoidal natural transformations as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left=40, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=40, "H"'] & \arrow[phantom, shift left=3ex, "\Downarrow\tau"] \arrow[phantom, shift right=3ex, "\Downarrow\sigma"] & \mathcal{C}', \end{tikzcd} \tag{B.32}$$

Then the vertical composite natural transformation $\sigma \cdot \tau : F \to H$ is symmetric monoidal.

Proof. The perimeters of the two diagrams below commute because every square commutes by assumption:

$$\begin{tikzcd} e' \arrow[r, "F_0"] \arrow[d, equal] & Fe \arrow[d, "\tau_e"] & Fa \otimes Fb \arrow[r, "F_2"] \arrow[d, "\tau_a \otimes \tau_b"'] & F(a \otimes b) \arrow[d, "\tau_{a\otimes b}"] \\ e' \arrow[r, "G_0"] \arrow[d, equal] & Ge \arrow[d, "\sigma_e"] & Ga \otimes Gb \arrow[r, "G_2"] \arrow[d, "\sigma_a \otimes \sigma_b"'] & G(a \otimes b) \arrow[d, "\sigma_{a\otimes b}"] \\ e' \arrow[r, "H_0"] & He & Ha \otimes Hb \arrow[r, "H_2"'] & H(a \otimes b). \end{tikzcd} \tag{B.33}$$

□

Lemma B.10. Let τ and τ' be symmetric monoidal natural transformations as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{C}' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\tau'"] & \mathcal{C}''. \end{tikzcd} \tag{B.34}$$

Then the horizontal composite natural transformation $\tau' \circ \tau : F'F \to G'G$ is symmetric monoidal.

Proof. The unit axiom for $\tau' \circ \tau$ amounts to the commutativity of the perimeter of

$$
\begin{tikzcd}
e'' \arrow[r, "F'_0"] \arrow[d, equal] \arrow[rr, bend left, "(F'F)_0"] \arrow[dr, phantom, "\boxed{1}"] & F'e' \arrow[r, "F'F_0"] \arrow[d, equal] \arrow[dr, phantom, "\boxed{2}"] & F'Fe \arrow[d, "F'\tau_e"] \arrow[dd, bend left, "(\tau'\circ\tau)_e"] \\
e'' \arrow[r, "F'_0"] \arrow[d, equal] \arrow[dr, phantom, "\boxed{3}"] & F'e' \arrow[r, "F'G_0"] \arrow[d, "\tau'_{e'}"] \arrow[dr, phantom, "\boxed{4}"] & F'Ge \arrow[d, "\tau_{Ge}"] \\
e'' \arrow[r, "G'_0"] \arrow[rr, bend right, "(G'G)_0"'] & G'e' \arrow[r, "G'G_0"] & G'Ge,
\end{tikzcd}
\qquad \text{(B.35)}
$$

where $\boxed{1}$ commutes trivially, $\boxed{2}$ commutes by applying F' to the unit axiom for τ, $\boxed{3}$ commutes by the unit axiom for τ', $\boxed{4}$ commutes by naturality of τ' relative to G_0, and the triangles commute by definitions.

The tensor axiom for $\tau' \circ \tau$ amounts to the commutativity of the perimeter of

$$
\begin{tikzcd}
F'Fa \otimes F'Fb \arrow[r, "F'_2"] \arrow[d, "F'\tau_a \otimes F'\tau_b"] \arrow[rr, bend left, "(F'F)_2"] \arrow[dd, bend right, "(\tau'\circ\tau)_a \otimes (\tau'\circ\tau)_b"'] \arrow[dr, phantom, "\boxed{1}"] & F'(Fa \otimes Fb) \arrow[r, "F'(F_2)"] \arrow[d, "F'(\tau_a \otimes \tau_b)"] \arrow[dr, phantom, "\boxed{2}"] & F'F(a \otimes b) \arrow[d, "F'\tau_{a\otimes b}"] \arrow[dd, bend left, "(\tau\circ\tau')_{a\otimes b}"] \\
F'Ga \otimes F'Gb \arrow[r, "F'_2"] \arrow[d, "\tau'_{Ga} \otimes \tau'_{Gb}"] \arrow[dr, phantom, "\boxed{3}"] & F'(Ga \otimes Gb) \arrow[r, "F'G_2"] \arrow[d, "\tau'_{Ga \otimes Gb}"] \arrow[dr, phantom, "\boxed{4}"] & F'G(a \otimes b) \arrow[d, "\tau'_{G(a\otimes b)}"] \\
G'aGa \otimes G'Gb \arrow[r, "G'_2"] \arrow[rr, bend right, "(G'G)_2"'] & G'(Ga \otimes Gb) \arrow[r, "G'G_2"] & G'G(a \otimes b)
\end{tikzcd}
\qquad \text{(B.36)}
$$

for every objects a and b in $\mathcal{C}$. Here $\boxed{1}$ commutes by naturality of F'_2, $\boxed{2}$ commutes by applying F' to the tensor axiom for τ, $\boxed{3}$ commutes by the tensor axiom for τ', $\boxed{4}$ commutes by naturality of τ' with respect to G_2, and the triangles commute by definitions. □

B.2. The 2-category of symmetric monoidal categories

The upshot of lemma B.9 is that the collection of symmetric monoidal functors $\mathrm{SM}(\mathcal{C},\mathcal{C}')$ can be made into a category $\underline{\mathrm{SM}}(\mathcal{C},\mathcal{C}')$ whose morphisms are symmetric monoidal natural transformations. Moreover, the last lemmas of the previous section show that the functor

$$\underline{\mathbf{Cat}}(\mathcal{Y},\mathcal{Z}) \times \underline{\mathbf{Cat}}(\mathcal{X},\mathcal{Y}) \xrightarrow{\circ} \underline{\mathbf{Cat}}(\mathcal{X},\mathcal{Z}), \tag{B.37}$$

defined in example A.3 yields a functor

$$\underline{\mathrm{SM}}(\mathcal{Y},\mathcal{Z}) \times \underline{\mathrm{SM}}(\mathcal{X},\mathcal{Y}) \xrightarrow{\circ} \underline{\mathrm{SM}}(\mathcal{X},\mathcal{Z}). \tag{B.38}$$

It is clear that the identity functor and the identity natural transformation are canonically symmetric monoidal. In conclusion, there is a 2-category $\underline{\mathrm{SM}}$ whose objects are categories, 1-cells are functors, and 2-cells are natural transformations – all of the symmetric monoidal variant.

Remark B.11. The category $\underline{\mathrm{SM}}(\mathcal{X},\mathcal{Y})$ is naturally a symmetric monoidal category, by taking tensor product pointwise. More precisely, given a pair of symmetric monoidal functors $F, G : \mathcal{X} \to \mathcal{Y}$, we let $F \otimes G$ be the functor defined by $(F \otimes G)(x) = F(x) \otimes G(x)$, for $x \in \mathcal{X}$. Similarly, given a pair of symmetric monoidal natural transformations $\tau : F \to G$ and $\tau' : F' \to G'$, we let $(\tau \otimes \tau')_x = \tau_x \otimes \tau'_x$. The associator, unitors and braiding are similarly given pointwise. It is then straightforward to verify that this gives a symmetric monoidal structure on $\underline{\mathrm{SM}}(\mathcal{X},\mathcal{Y})$, using that $\mathcal{Y}$ is a symmetric monoidal category.

B.3. Invertible objects

Let x be an object of a symmetric monoidal category $\mathcal{C}$. Seeing x as a functor $x : * \to \mathcal{C}$, we denote by $x \otimes -$ the following composite functor:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\cong"] \arrow[rrr, bend right, "x\otimes-"'] & \mathcal{C} \times * \arrow[r, "1\times x"] & \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] & \mathcal{C}. \end{tikzcd} \tag{B.39}$$

Definition B.12. The object x is said to be *invertible* if there exists an object y in $\mathcal{C}$ and an isomorphism $\delta : x \otimes y \cong e$.

Lemma B.13. If x is invertible, then the functor $x \otimes -$ is an equivalence of categories.

Proof. Assume $\delta : x \otimes y \cong e$. We show that the functor $y \otimes -$ is an "inverse" to the functor $x \otimes -$. Given an object z in $\mathcal{C}$, we have an isomorphism

$$x \otimes (y \otimes z) \xrightarrow[\cong]{\alpha} (x \otimes y) \otimes z \xrightarrow[\cong]{\delta \otimes 1} e \otimes z \xrightarrow[\cong]{\lambda} z. \tag{B.40}$$

This isomorphism is natural in z, since the perimeter of the following diagram commutes:

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "1\otimes(1\otimes f)"] \arrow[d, "\alpha"', "\cong"] & x \otimes (y \otimes w) \arrow[d, "\cong"', "\alpha"] \\
(x \otimes y) \otimes z \arrow[r, "(1\otimes 1)\otimes f"] \arrow[d, "\delta\otimes 1"', "\cong"] & (x \otimes y) \otimes w \arrow[d, "\cong"', "\delta\otimes 1"] \\
e \otimes z \arrow[r, "1\otimes f"] \arrow[d, "\lambda"', "\cong"] & e \otimes w \arrow[d, "\cong"', "\lambda"] \\
z \arrow[r, "1"'] & z.
\end{tikzcd} \tag{B.41}$$

Here the top and bottom squares commute by naturality of α and λ, and the middle square commutes by the interchange law. □

C. Categorical miscellanea

We briefly record some elementary results about the algebra of morphisms in a category. Let us fix the following data of objects and morphisms in some category:

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d. \tag{C.1}$$

Lemma C.1. If g has both a pre-inverse s and a post-inverse t, i.e. the following diagram commutes:

$$\begin{tikzcd} c \arrow[r, "s"] \arrow[rr, bend right, "1_c"'] & b \arrow[r, "g"] \arrow[rr, bend left, "1_b"] & c \arrow[r, "t"] & b, \end{tikzcd} \tag{C.2}$$

then $s = t$ and g is an isomorphism.

Proof. $t = t1_c = tgs = 1_b s = s$, hence g is an isomorphism with inverse $s = t$. □

The following result is known as the *2-out-of-6 property* for isomorphisms:

Proposition C.2. If gf and hg are isomorphisms, i.e. there is a diagram of the form:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend right, "\cong"'] & b \arrow[r, "g"] \arrow[rr, bend left, "\cong"] & c \arrow[r, "h"] & d, \end{tikzcd} \tag{C.3}$$

then f, g, h are isomorphisms.

Proof. We first show that g is an isomorphism. Let $s = f(gf)^{-1}$ and $t = (hg)^{-1}h$. Then $gs = gf(gf)^{-1} = 1_c$ and $tg = (hg)^{-1}hg = 1_b$, hence $s = t$ and g is an isomorphism by lemma C.1.

Next we show that f is an isomorphism. We see that f has pre-inverse $(gf)^{-1}g$, since

$$f(gf)^{-1}g = (1_b)(f(gf)^{-1}g) = (g^{-1}g)(f(gf)^{-1}g) = g^{-1}(gf(gf)^{-1})g = g^{-1}g = 1_b. \tag{C.4}$$

Moreover, f has a post-inverse $(gf)^{-1}g$, since $(gf)^{-1}gf = 1_a$. Thus f is an isomorphism, and a symmetric argument shows that h is an isomorphism. □

Corollary C.3. Let $i : x \to y$ and $j : y \to x$ be morphisms in some category, such that both composites ij and ji are isomorphisms. Then i and j are isomorphisms.

Corollary C.4. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor sending endomorphisms to isomorphisms. Suppose that for every morphism $x \to y$ in $\mathcal{C}$ there exists a morphism $y \to x$ in $\mathcal{C}$. Then the functor F sends every morphism to an isomorphism.

Proof. Let $i : x \to y$ be an arbitrary morphism. Then by assumption there exists $j : y \to x$. Now ij and ji are endomorphisms, hence $Fij = FiFj$ and $Fji = FjFi$ are isomorphisms. By corollary C.3, we conclude that Fi and Fj are isomorphisms. □

Bibliography

[1] Karol Borsuk. *Theory of retracts*. Vol. 44. Państwowe Wydawn. Naukowe, 1967.

[2] Pavel Etingof et al. *Tensor categories*. Vol. 205. American Mathematical Soc., 2016.

[3] Jean Gallier and Dianna Xu. *A guide to the classification theorem for compact surfaces*. Springer Science & Business Media, 2013.

[4] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.

[5] R Juer and U Tillmann. "Localisations of cobordism categories and invertible TFTs in dimension two". In: *Homology, Homotopy and Applications, 15(2), 2013, 1-31* (2013).

[6] Daniel Quillen. "Higher algebraic K-theory: I". In: *Higher K-theories*. Springer, 1973, pp. 85–147.

[7] Maja D. Rabrenović. "Classifying spaces of monoids—applications in geometric combinatorics". In: *Publ. Inst. Math. (Beograd) (N.S.)* 77(91) (2005), pp. 21–27. ISSN: 0350-1302. DOI: 10.2298/PIM0591021R. URL: https://doi.org/10.2298/PIM0591021R.

[8] Ulrike Tillmann. "The classifying space of the 1 + 1 dimensional cobordism category." In: *Journal fur die reine und angewandte Mathematik* 479 (1996), pp. 67–76. URL: http://eudml.org/doc/153854.